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Nonlinear Dynamics and Stability of a Microbeam Array Subject to Parametric Excitation

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Nonlinear Dynamics and Stability of a Microbeam Array Subject to Parametric Excitation

RESEARCH THESIS

In Partial Fulfillment of The Requirements for the Degree of Master of Science in Mechanical Engineering

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Abstract

In the past decade, arrays of micro-resonators have been successfully used as storage devices and for mapping of surfaces via scanning probe microscopy (SPM) and as coupled microbeam arrays which exhibit complex electrically tunable collective response. The performance of both example arrays are governed by nonlinear effects. The SPM array can at specific operating conditions, yield spurious random-like surface maps of periodic structures, whereas the complex response of the tunable array is not yet completely understood.

We have investigated the nonlinear dynamics of a three element microbeam array that is subject to electrodynamic parametric excitation. We derived a theoretical continuum based model for the array which incorporates linear viscoelastic material properties and a geometric nonlinearity truncated to cubic order. Solution of the reduced order nonlinear modal dynamical system near its principle parametric resonance is obtained via multiple scale asymptotics. The theoretical analysis is verified numerically and reveals a complex bifurcation structure which includes coexisting stable periodic in-phase and out-of-phase dynamics. We manufactured and tested a three element array of gold microbeams, and present a qualitative comparison between theory and experiments. A relatively good agreement between theory and experiment has been obtained for the case of a single beam, demonstrating the importance of modeling nonlinear damping which is essential for obtaining bounded response for arrays that are excited parametrically.
List of Symbols

Chapter 2

$\sigma$  stress
$E$  elastic modulus
$\epsilon$  strain
$D$  viscosity
$U_n$  longitudinal beam-string displacement of the $n$–th beam
$W_n$  transverse beam-string displacement of the $n$–th beam
$t$  time coordinate
$\varsigma$  position coordinate
$X_{coordinate}$  derivative of $X$ with respect to the subscripted coordinate
$Q$  electrodynamic force
$\rho$  density
$A$  cross sectional area
$N$  pretension
$D$  viscoelastic damping
$I$  second moment of inertia
$U$  capacitive energy
$\varepsilon_0$  permittivity of free space ($\varepsilon_0 = 8.854 \times 10^{-12} \, \text{F/m}$)
$B$  beam width
$V_{AC}$  AC voltage
$|V_{AC}|$  AC voltage amplitude
$\omega_{AC}$  AC voltage frequency
\( G_n \) gap width between beam \( n \) and beam \( n-1 \)
\( G_{n+1} \) gap width between beam \( n \) and beam \( n+1 \)
\( G \) symmetric gap width between neighboring beams
\( \bar{G} \) average gap width between neighboring beams \( (\frac{G_n + G_{n+1}}{2}) \)
\( L \) beam length
\( u_n \) scaled longitudinal beam-string displacement of the \( n \)-th beam \( (u_n = LU_n) \)
\( w_n \) scaled transverse beam-string displacement of the \( n \)-th beam \( (w_n = LW_n) \)
\( \omega_s \) time scaling constant \( (\omega_s = \sqrt{\frac{N}{\rho AL^2}}) \)
\( \tau \) scaled time coordinate \( (\tau = \sqrt{\frac{\rho AL^2 t}{N}}) \)
\( s \) scaled position coordinate \( (s = L\varsigma) \)
\( \bar{\alpha} \) scaled pretension parameter \( (\bar{\alpha} = \frac{E\alpha}{N}) \)
\( \bar{\delta} \) scaled viscoelastic damping parameter \( (\bar{\delta} = \frac{D}{L} \sqrt{\frac{\bar{\alpha}}{\rho E}}) \)
\( \bar{\xi} \) scaled bending parameter \( (\bar{\xi} = \frac{EI}{NL^2}) \)
\( \bar{\beta} \) scaled linear damping parameter \( (\bar{\beta} = \frac{DI}{AL^3} \sqrt{\frac{\bar{\alpha}}{\rho E}}) \)
\( r \) radius of gyration \( (r = \sqrt{\frac{I}{A}}) \)
\( \hat{Q} \) scaled electrodynamic force \( (\hat{Q} = \frac{LQ}{N}) \)
\( \eta \) scaled AC voltage amplitude \( (\eta = \frac{\varepsilon_0 B|V_{AC}|^2}{4LN}) \)
\( \hat{\Omega} \) scaled AC voltage frequenct \( (\hat{\Omega} = \frac{\omega_{AC}}{\omega_s}) \)
\( \gamma_n \) scaled gap width between \( n \)th and \( n-1 \)th beams \( (\gamma_n = \frac{G_n}{L}) \)
\( \gamma_{n+1} \) scaled gap width between \( n \)th and \( n+1 \)th beams \( (\gamma_{n+1} = \frac{G_{n+1}}{L}) \)
\( \gamma \) scaled average gap width between neighboring beams \( (\gamma = \frac{\bar{G}}{L}) \)
\( \eta_{PL} \) scaled AC voltage amplitude at which pull in occurs
\( \Gamma_i \) Taylor expansion constants \( (\Gamma_i = (1 + i\frac{1}{\gamma_{n+1}})) \)
\( q_n, q_{n1} \) temporal mode shape of the transverse vibration of a string
\( \phi_1(s) \) primary mode shape \( (\text{harmonic string mode, } \phi_1(s) = \sqrt{2}\sin(\pi s)) \)
\( I_j \) Galerkin integrals
\( (I_0 = \frac{2\sqrt{2}}{\pi}, I_1 = 1, I_2 = \frac{8\sqrt{2}}{3\pi}, I_3 = \frac{3}{2}, I_4 = -\pi^2, I_5 = \pi^2, I_6 = \pi^4) \)
\( \bar{\phi} \) displacement magnitude at the center of the string \((\bar{\phi} = \sqrt{2})\)

\( x_n \) scaled time function of the transverse vibration of a string \((x_n = \frac{\bar{\phi} q_n}{\gamma})\)

\( x_{ni} \) scaled time function of the transverse vibration of the \(i\)th beam in a beam array \((x_{ni} = \frac{\bar{\phi} q_{ni}}{\gamma})\)

\( \bar{\omega}_1 \) unperturbed natural frequency \((\bar{\omega}_1 = \pi \sqrt{1 + \xi \pi^2})\)

\( \alpha \) nonlinear stiffness coefficient \((\alpha = \frac{\pi^2 EA \gamma^2}{4N(1+\xi \pi^2)})\)

\( \beta \) linear damping coefficient \((\beta = \frac{\pi^3 DI}{L^3 \sqrt{NA \rho (1+\xi \pi^2)}})\)

\( Q \) quality factor \((\frac{\omega_1}{\beta})\)

\( \delta \) nonlinear damping coefficient \((\delta = \frac{\pi^3 D \sqrt{A \gamma^2}}{2L \sqrt{NA \rho (1+\xi \pi^2)}})\)

\( \hat{\eta}_i \) forcing parameters \((\hat{\eta}_i = \frac{\Gamma_i I_i}{\omega_1^2} (\frac{\pi}{2})^{(i-1)})\)

\( \Omega \) forcing frequency \((\Omega = \frac{\hat{\Omega}}{\omega_1})\)

\( \nu \) forcing amplitude coefficient \((\nu = \frac{\eta}{\gamma \omega_1^2})\)

**Chapter 3**

\( J \) Jacobian matrix

\( J_6 \) 6x6 Jacobian matrix

\( K \) linear stiffness matrix

\( M \) mass matrix

\( \omega_{ni} \) natural frequency of the \(n\)th beam

**Chapter 4**

\( \varepsilon \) small mathematical parameter that separates order equations in the asymptotic solution

\( T_i \) time scales \((T_i = \varepsilon^i \tau)\)

\( D_i^j \) \(j\)th order partial time derivative with respect to \(T_i\)

\( D_2 \) 1st order slow time derivative

\( \cdot \) 1st partial derivative with respect to \(T_2\) \((D_2)\)

\( O(\varepsilon) \) expression of the order \(\varepsilon\)

\( \hat{\beta} \) scaled linear damping parameter \((\hat{\beta} = \frac{\beta}{\varepsilon^2})\)

\( \hat{\nu} \) scaled forcing parameter \((\hat{\nu} = \frac{\nu}{\varepsilon^2})\)

\( A \) first order equation solution’s amplitude
$A_i$ amplitude of first order equation of $i$th beam’s solution

$\bar{A}$ first order equation solution’s complex conjugate amplitude

$\bar{A}_i$ complex conjugate amplitude of first order equation of $i$th beam’s solution

$\varepsilon\sigma$ detuning from principle primary resonance ($\varepsilon\sigma = \Omega - 1$)

$a$ order equation polar solution’s amplitude

$a_i$ order equation polar solution of $i$th beam’s amplitude

$\theta$ order equation polar solution’s phase

$\theta_i$ phase of polar solution of $i$th beam’s order equation

$\phi$ evolution equation phase ($\phi = \sigma T_2 - \theta$)

$\phi_i$ phase of $i$th beam evolution equation ($\phi_i = \sigma T_2 - \theta_i$)

$a_i$ coefficients used in definition of order equation solution’s phase that are functions of the parameters $\alpha, \beta, \delta, \nu$

$\Omega|_{\varepsilon a=0}$ frequency where nontrivial solution to the order equation branches off from the trivial solution

$BW$ bandwidth of the nontrivial solution to the order equation

$BW_3$ bandwidth of the nontrivial solutions to the three beam order equations

$c_i$ coefficients that are functions of the parameters $\alpha, \beta, \delta, \nu$

$\Omega_{amax}$ frequency where the amplitude of the solution to the order equation reaches its maximum

$(\varepsilon a)_{max}$ maximum amplitude of the solution to the order equation

$J_{polar}$ Jacobian matrix of polar order equations

$\lambda$ eigenvalue

$\Omega_{saddle}$ frequency of order equation solution’s saddle point

$(\varepsilon a)_{saddle}$ amplitude of the solution to the order equation where saddle point occurs

$p_i, q_i$ Cartesian solution coordinates of order equation

$p_i, q_i$ Cartesian solution coordinates of $i$th beam order equation

$J_{cartesian}$ Jacobian matrix of Cartesian order equations

$\hat{\delta}$ scaled nonlinear damping parameter ($\hat{\delta} = \frac{\delta}{8}$)

$\hat{\alpha}$ scaled nonlinear stiffness parameter ($\hat{\alpha} = \frac{3\alpha}{8}$)

$\hat{\beta}$ scaled linear damping parameter ($\hat{\beta} = \frac{\beta}{2}$)
$J_{66}$  
6x6 Jacobian matrix of Cartesian order equations

$b_i$  
coefficients that are functions of the parameters $\Omega, \beta, \nu$ used in the characteristic equation of the three beam Cartesian order equations

$\Omega_{b_6=0}$  
frequency at which the zero degree coefficient of the characteristic equation of the three beam Cartesian order equations is equal to zero as the function of the parameters $\beta$ and $\nu$

Chapter 6

$H$  
beam height

$f_i$  
$i$th mode resonance frequency of single beam ($f_i = \frac{\omega_i}{2\pi}$)

$B_{Au,Pd,CR}$  
width of beam layer composed of gold, palladium, or chromium, respectively

$MP_{Au,Pd,CR}$  
material property (density or elastic modulus) of gold, palladium, or chromium, respectively

$E_{LOM}$  
elastic modulus of beam calculated using the law of mixtures

$\rho_{LOM}$  
density of beam calculated using the law of mixtures

$\frac{E_{DCmeas}}{\rho_{DCmeas}}$  
ratio between density and elastic modulus of beam calculated from measurements taken under the application of DC voltage

$BW_i$  
possible response bandwidth based off of experimental data

$\Omega_{max}$  
frequency where the measured response amplitude reaches its maximum based off of experiments done on a single beam under the application of an AC voltage

$|\varepsilon_a|_{max}$  
maximum measured response amplitude based off of experiments done on a single beam under the application of an AC voltage

$\Omega^{*}_{saddle}$  
frequency of saddle node of measured response to parametric excitation

$|\varepsilon_a|_{saddle}$  
amplitude of measured response of single beam under parametric excitation where saddle point occurs

$T_{data}$  
hypothesized linear transfer function used to obtain arbitrary units of experimental data

$a*$  
slope of $T_{data}$

$b$  
intercept of $T_{data}$

$T_{max}$  
maximum response of experimental data in arbitrary units

$T_{saddle}$  
response amplitude of experimental data in arbitrary units at saddle point

$|x|$  
inverse transfer function of $T_{data}$
Appendices

\[ T_{1i} \] hypothesized linear transfer function for \( i \)th possible bandwidth, \( BW_i \), used to obtain arbitrary units of experimental data

\[ a_{*i} \] slope of \( T_{12} \)

\[ b_i \] intercept of \( T_{1i} \)

\[ T_{meas} \] measured response of experimental data in arbitrary units

\[ |x|_i \] inverse transfer function of \( T_{1i} \)

\[ T_{*2} \] hypothesized quadratic transfer function used to obtain arbitrary units of experimental data

\[ c, d, e \] coefficients of \( T_{*2} \)

\[ |x|_q \] inverse transfer function of \( T_{*2} \)

\[ T_{2i} \] hypothesized quadratic transfer function for \( i \)th possible bandwidth, \( BW_i \), used to obtain arbitrary units of experimental data

\[ |x|_{2i} \] inverse transfer function of \( T_{2i} \)
Chapter 1

Introduction

1.1 Motivation

Micro- and nano- resonators are small flexible structures (beams, wires, fibers, plates, nanotubes, etc.) that naturally oscillate with greater amplitude at their resonance frequencies. The oscillations can be induced by electrodynamic, optical, mechanical, or magnetic fields.

These oscillating elements can be externally or parametrically excited. External excitation results from a constant excitation that is not dependent on the dependent variables of the problem. Parametric excitation occurs when one of the parameters is allowed to vary with time [41, 37, 34, 5]. The first time parametric excitation was observed was by Michael Faraday in 1831 [16]. He noticed that surface oscillations were being generated in a bucket of water at half the frequency with which vertical waves were being excited in that bucket. In 1859 Melde was the first to create parametric oscillations [46]. He used a tuning fork to periodically vary the tension in a string at twice its resonance frequency. The first time that this behavior was mathematically modeled was by Mathieu in 1868 [32]. In 1883 Lord Rayleigh gave mathematical justification to the behavior observed by Faraday. In 1887 he provided mathematical analysis for Melde’s work [38].

Micro- and nano- resonators can self-induce oscillations without the application of any external field, due to the photothermal effect [21]. When a constant wave laser beam irradiates a microbeam, some of the incident light is absorbed by the beam and the rest travels past the beam towards a reflective surface located beneath the beam. The light is reflected by the surface, and interferes with the original light beam which is propagating towards the surface to produce a standing wave between the beam and the reflective surface. The beam absorbs the light energy of the standing wave and converts it to heat.
energy. This results in the photothermal effect. Heat which spreads along the thickness of the glass plate generates a thermoelastic bending moment. Consequently the beam deflects, and therefore the intensity of the light impinging on the absorption layer varies according to the position of the layer. The absorption of light energy and its conversion to heat occur periodically, culminating in beam vibration at one of the cantilever’s resonance frequencies beyond a critical threshold. These self-excited oscillations are used to generate waves of a specific frequency or to select a particular frequency from a signal. They can be used to measure physical parameters such as vibration, temperature, force, stress, acceleration, pressure, mass, and magnetic fields since the microresonator natural frequency is a function of these parameters.

The fabrication of many micro- or nano- resonators in the form an array [23, 50, 2, 11] can enable interaction of individual elements with their neighbors to enhance their performance, as both the magnitude and bandwidth of the response are increased. The array behaves in a way that individual elements can not [7]. These arrays have a wide variety of applications which include optomechanical signal processing [7] and signal mixing [15], high-speed imaging [39, 4, 43] with atomic force microscopy [36, 17], laser scanning [49], piezoresistive sensing [31] and Scanning Electrochemical Microscopy [17], parallel lithography [28, 22], parallel data storage with high data density [47, 11, 26], biosensing using devices such as artificial noses that detect analyte vapors [3], mass sensing [10, 23], and protein printing for cell adhesion [44]. It has been shown that considering three beams from the end of a larger a microbeam array is a good representative of the general behavior of the array in terms of its linear dynamics [51]. Further study is necessary however to determine how well it represents the nonlinear dynamics.

The dynamic response of an array of coupled oscillators can be governed by nonlinear effects [7, 30, 6, 9, 14, 23]. The study of these arrays has been mainly experimental [7, 36, 44]. Documented theoretical models consist mainly of finite element [44] and finite difference models [48] and lumped-mass systems [6, 13, 30] which qualitatively reproduce measured array response (in and out-of-phase periodic response), but do not take into account boundary conditions or dispersion relationships or incorporate true elastic behavior, where linear and nonlinear viscoelastic parameters are related. To date, there has not been any quantitative comparison between any of the aforementioned theoretical and experimental investigations. Recently, investigations of the intrinsic energy localization of specific array modes [14, 25, 45], of a stability analysis based on Lyapunov exponents [27], and of internal resonances [19, 20, 18] have been performed. A continuum mechanics model [19, 20, 18] was derived for a multi-element array and analyzed analytically and numerically for two and three micro-beam systems, respectively.
1.2 Survey of Related Literature

BUKS & ROUKES [7] studied the behavior of an electrically tunable grating array of suspended, doubly-clamped, electrostatically coupled, parametrically excited, gold beams. Their results showed that the collective response of the array is an oscillatory function of frequency that has a relatively large magnitude close to the lower limit of the frequency band which in a sweep up of the frequency gradually increased and abruptly decreased. The array also responded at frequencies above the expected top edge of the band. Note that since the data acquisition in this experiment was obtained by taking the mean optical measure of the intensity sampled by a spectrum analyzer, and does not include a time series, there is no evaluation of possible coexisting bistable solutions or of possible quasiperiodic or chaotic solutions of individual elements of the array due to internal resonances and other strongly nonlinear mechanisms.

LIFSCHITZ & CROSS [30] derived a theoretical model of the BUKS/ROUKES system that consists of a set of coupled Duffing-type lumped-mass equations of motion for an array excited at its principal parametric resonance and includes the effects of nonlinear elasticity and dissipation. They succeeded in qualitatively reproducing some of the features observed in the experiment such as the jump behavior of the response and its responding above the expected maximum frequency. They also discovered coexisting stable and unstable periodic solutions for simultaneous large external (DC) excitation and small parametric (AC) excitation. However, their analysis did not reveal the existence of any quasiperiodic energy transfer between neighboring beams.

GUTSCHMIDT & GOTTLIEB [18, 20, 19] modeled a continuum initial-boundary-value problem (IBVP) of a doubly-clamped microbeam array excited at several DC biases and periodic AC voltages. They used a parallel plate model for the electrostatic force and include a restoring force that consists of linear and nonlinear stiffness terms as well as residual stress, and a damping force that is composed of both a linear viscous and a viscoelastic term. For a zero (and near-zero) DC bias the natural frequencies of the array were identical and thus, the system was excited at its one-to-one internal resonance [18]. For DC-voltage inputs near the systems first pull-in instability several three-to-one internal and combinational resonances were identified. In both cases (low and high DC bias), analytical and numerical analyses of a two beam array revealed multiple coexisting stable and unstable, periodic and aperiodic solutions. Quasiperiodic responses were found in a numerical analysis of a three beam electrostatically excited array whose electrostatic force was based off of the parallel plate model [20]. The differences between lumped-mass and continuum based models include consistent incorporation of true system boundary
conditions and a model based relationship between linear material properties and nonlinear system parameters. Consequently, a calibrated IBVP-based dynamical system will enable accurate prediction of dynamic system response beyond the validity of a phenomenological lumped-mass model.

In Chapter 6 results will be presented that suggest the presence of self-excited oscillations. A number of studies [1, 21, 35] have been done on self-excited vibrations in micromechanical resonators. A self-excited system is one that in the absence of external, modulated forcing will vibrate at a steady amplitude, i.e.- undergoes limit cycle oscillations. In micro-systems, thermally driven oscillations can occur when a beam or similar structure moves within an interference pattern. The forces that drive the oscillation can be due to thermal bending, thermal expansion of nonplanar devices, thermal strain driven stiffness modulation, photon pressure, and opto-electronic strain. When laser intensity is low, the beam bends statically. When it increases, the beam begins to vibrate, i.e.- to undergo limit cycle oscillations.

In optomechanical systems, the mechanical Q factor is enhanced since the mechanical degree of freedom extracts energy provided by the optical radiation. This brings about a parametric instability, which drives the system into a state of self-sustained oscillations. Nonlinear behavior such as mode jump, hysteresis, and amplitude limit have been observed in these oscillations. It was found that the self-excited vibration has an incident light power threshold. This threshold changes based on the direction of changing the intensity. The quality-factor affects the laser power threshold. It was postulated by AUBIN ET AL. [1] that $P_{Hopf} \sim \frac{1}{Q^2\pi}$. They also find that a certain threshold that depends on the device properties and the placement of the beam, the device jumps into motion in a Hopf-type bifurcation. It is estimated that the peak-to-peak amplitude of motion can be as high as half the wavelength of the laser light. It was also found that the vibration amplitude varies nonlinearly with the increase of the laser power. ([21])

The nonlinear dynamics of this system can become significantly influential, leading to an intricate attractor diagram. In [35] through analysis of an attractor diagram, it was discovered that at large optical power there is an onset of multi-mode dynamics, with two mechanical modes of the cantilever participating in the radiation-driven self-sustained oscillations.

In addition, the simultaneous excitation of several mechanical modes of the cantilever leads to coupled nonlinear dynamics. Since resonance frequency is sensitive to temperature, liquid density, and pressure, this is an effective way to conduct spectroscopic studies and non-destructive inspection. Optical vibration-sensors are immune to electromagnetic
noise and are intrinsically compatible with digital systems.

1.3 Objectives

In order to resolve the open issues noted previously, the main goal of this research is to investigate theoretically and experimentally the nonlinear dynamics and orbital stability of a three-element micro-beam array that is subject to parametric excitation. This is achieved by the following specific objectives:

i the derivation of a continuum mechanics based model for a general $n$-beam array that takes into account nonlinear effects due to nearest neighbor interaction, describes true spatio-temporal dynamics and includes linear viscoelastic material properties and nonlinear geometry;

ii the reduction of this model to a three degree-of-freedom modal dynamical system which will enable a consistent asymptotic and numerical analysis;

iii the solution of the dynamical system via multiple scale asymptotics which will enable construction of a comprehensive bifurcation structure that describes coexisting stable/unstable, periodic/aperiodic solutions;

iv the numerical validation of the asymptotic solutions and investigation of the strongly nonlinear regime of operation;

v the manufacturing and testing of an experimental three-beam array and comparison of experimental results with those obtained from the theoretical model.

1.4 Outline of the Thesis

The rest of this thesis is structured as follows. Chapter 2 presents the derivation of the modal equations from the initial boundary value problem. Chapter 3 examines the stability of the fixed points. Chapter 4 derives the slowly varying evolution equations by the multiple scales method. Chapter 5 shows and compares the results of numerical integration of the modal differential equations as well as the multiple scales equations and the analytical equilibrium solution of the polar evolution equations for a single beam. Chapter 6 presents and analyzes the experimental results and compares them with the theory. Finally, concluding remarks and potential extensions of this study are presented in Chapter 7.
Chapter 2

Model

In this chapter we derive the initial-boundary-value problem (IBVP) for the three element array, and reduce it to a modal dynamical system for a three element array.

2.1 Initial-Boundary-Value Problem

We base our model derivation on the equations for a planar weakly nonlinear pretensioned (N) visco-elastic beam-string which incorporates a Voigt-Kelvin ($\sigma = E\epsilon + D\dot{\epsilon}$) constitutive relationship [29]:

\begin{align}
\rho A U_{ntt} - \left[NU_{n\varsigma} + EA(U_{n\varsigma} + \frac{1}{2}W^2_{n\varsigma}) + DA(U_{nt\varsigma} + W_{n\varsigma}W_{nt\varsigma})\right]_{\varsigma} &= Q_{un} \quad (2.1.1) \\
\rho A W_{ntt} - \left[NW_{n\varsigma} + EAW_{n\varsigma}(U_{n\varsigma} + \frac{1}{2}W^2_{n\varsigma}) + DAW_{n\varsigma}(U_{nt\varsigma} + W_{n\varsigma}W_{nt\varsigma}) \right. \\
& \left. -(EIW_{n\varsigma\varsigma\varsigma} + DIW_{nt\varsigma\varsigma})\right]_{\varsigma} = Q_{wn},
\end{align}

where $U_n(\varsigma,t)$ and $W_n(\varsigma,t)$ are the longitudinal and transverse components of an elastic field, respectively. Note that $U_0 = U_{n+1} = W_0 = W_{n+1} = 0$. The generalized force components ($Q_{un,wn}$) are due to external electrodynamic actuation and they are of the form:

\begin{equation}
Q_{un,wn} = -\left(\frac{\partial U}{\partial (w, u)_{n-1}} - 2\frac{\partial U}{\partial (w, u)_n} + \frac{\partial U}{\partial (w, u)_{n+1}}\right), \quad (2.1.3)
\end{equation}

where $U$ is the capacitive energy.
We assumed fixed boundary conditions, namely:

\[
U_n(0, t) = U_n(L, t) = W_n(0, t) = W_n(L, t) = W_{nς}(0, t) = W_{nς}(L, t) = 0, \quad (2.1.4)
\]

and nontrivial initial conditions.

Eq.s (2.1.1)-(2.1.4) constitute the array initial-boundary-value problem.

Note that for a parallel plate approximation, the longitudinal force component is negligible \((Q_{un} = 0)\) and the transverse component for an asymmetric configuration is:

\[
Q_{wn} = \left( \frac{c_0 BV_{AC}^2}{2} \right) \left[ \frac{1}{(G_n - w_n + w_{n-1})^2} - \frac{1}{(G_{n+1} + w_n - w_{n+1})^2} \right], \quad v_{AC} = |v_{AC}| \cos (\omega_{AC} t) \quad (2.1.5)
\]

Furthermore, we assume a symmetric configuration consists of equal gaps from both upper and lower electrodes (or the previous beam) to the beam surface \((G_n = G_{n+1} = G)\).

We rescale the elastic field components \((U_n, W_n)\) and material coordinate \((ς)\) by the beam length \((L)\) and time by the pretension scale \((\sqrt{\frac{\rho A L^2}{N}})\) to yield:

\[
\begin{align*}
\hat{u}_{nττ} - \left[ u_{ns} + \hat{α}(u_{ns} + \frac{1}{2} w^2_{ns}) + \hat{δ}(w_{nτs} + w_{ns} w_{nτs}) \right]_s &= \hat{Q}_{un} \quad (2.1.6) \\
\hat{w}_{nττ} - \left[ w_{ns} + \hat{α} w_{ns}(u_{ns} + \frac{1}{2} w^2_{ns}) + \hat{δ} w_{ns}(u_{nτs} + w_{ns} w_{nτs}) - \hat{ξ} w_{ns} + \hat{δ} w_{nτss} \right]_s &= \hat{Q}_{wn}, \quad (2.1.7)
\end{align*}
\]

where the force components are normalized by the pretension \((\hat{Q}_{un, wn} = \frac{LQ_{un, wn}}{N})\) and the non-dimensional parameters include the effects of weak bending \((\hat{ξ} < 1)\), a strong nonlinear pretension \((\hat{α} > 1)\), a small slenderness ratio \((\hat{β} < 1, \text{as } \frac{r}{L} << 1 \text{ where } r = \sqrt{\frac{I}{A}} \text{ is the beam-string radius of gyration})\), and finite viscoelastic damping \((\hat{δ})\):

\[
\begin{align*}
\hat{ξ} &= \frac{EI}{NL^2}, \quad \hat{α} = \frac{EA}{N}, \quad \hat{β} = \frac{DI}{AL^3} \sqrt{\frac{\hat{α}}{ρE}}, \quad \hat{δ} = \frac{D}{L} \sqrt{\frac{\hat{α}}{ρE}} \quad (2.1.8)
\end{align*}
\]

Note that \(\sqrt{\hat{α}}\) defines the ratio between the longitudinal and transverse wave speeds in the beam-string element. The rescaled parallel plate approximation is thus:

\[
\hat{Q}_{wn} = η(1 + \cos (2Ωτ)) \left[ \frac{1}{(γ_n - w_n + w_{n-1})^2} - \frac{1}{(γ_{n+1} + w_n - w_{n+1})^2} \right] \quad (2.1.9)
\]
which is governed by a non-dimensional amplitude $(\eta)$, frequency $(\hat{\Omega})$, and gap $(\gamma_{n,n+1})$ parameters:

$$\gamma_n = \frac{G_n}{L}, \quad \eta = \frac{\varepsilon_0 B |V_{AC}|^2}{4LN}, \quad \hat{\Omega} = \frac{\omega_{AC}}{\omega_s}, \quad \omega_s^2 = \frac{N}{\rho AL^2}. \tag{2.1.10}$$

We note that close to primary resonance of the transverse mode, the longitudinal velocity and inertia can be neglected to yield a simple spatial relationship between the transverse and longitudinal derivatives ($\tilde{\alpha} >> 1$, $Q_{un} = 0$). (The significance of $\tilde{\alpha} >> 1$ is that the longitudinal natural frequency will be much larger than the transverse natural frequency, the longitudinal inertia and damping will have negligible influence, and therefore their terms can be ignored.) We note that a quasisteady approach can be employed close to primary resonance of primary mode. Since the longitudinal displacement terms are of an order of the square of the transverse displacement terms, their contribution is to the transient term. Therefore, when analyzing the steady state behavior, the transverse displacement is significant. The longitudinal and transverse velocities as well as the longitudinal inertia can be neglected. Substituting these assumptions into Eq. (2.1.6) yields:

$$- \left[ u_{ns} + \tilde{\alpha} (u_{ns} + \frac{1}{2} w_{ns}^2) + \tilde{\delta} (w_{ns} w_{nrs}) \right]_s = 0 \tag{2.1.11}$$

Incorporating fixed boundary conditions $(u_n(0, \tau) = u_n(1, \tau) = 0)$ (and taking into account that for $\tilde{\alpha} >> 1$, $\frac{\tilde{\alpha}}{1+\tilde{\alpha}} \approx 1$) enables integration of the resulting relationship of Eq. (2.1.11) to yield:

$$u_{ns} = - \frac{1}{2} w_{ns}^2 + c_{n1}(\tau), \quad c_{n1} = \frac{1}{2} \int_0^1 w_{ns}^2 ds, \quad c_{n1\tau} = \int_0^1 w_{ns} w_{nrs} ds \tag{2.1.12}$$

Thus, the weakly nonlinear IBVP (previously defined) resulting from the substitution of Eq. (2.1.12) into Eq. (2.1.8) is an integro-differential equation for the transverse mode:

$$w_{n\tau\tau} - w_{nss}[1 + \tilde{\alpha} c_{n1}(\tau) + \tilde{\delta} c_{n1\tau}(\tau)] + \tilde{\xi} w_{nssss} + \tilde{\beta} w_{nrrssss} = \hat{Q}_{wn}. \tag{2.1.13}$$

We can further expand the electrodynamic force about its stable equilibrium ($w_n = 0$, $\eta < \eta_{PT}$) to yield:
\[ Q_{kn} = \eta(1 + \cos(2\Omega\tau)) \sum_{i=0}^{3} \Gamma_{in}(w_n - w_{n-1})^i - (-1)^i \Gamma_{in+1}(w_n - w_{n+1})^i, \]
\[ \Gamma_m = (1 + i) \frac{1}{\gamma_{m+2}}, \Gamma_{m+1} = (1 + i) \frac{1}{\gamma_{m+2}}. \]  

(2.1.14)

Note that for equal gaps \( (\gamma_n = \gamma_{n+1} = \gamma, \Gamma_m = \Gamma_{m+1} = \Gamma_i) \) the bias term cancels out and becomes zero \( (\Gamma_0 - \Gamma_0 = 0) \) and the quadratic term is completely dependent on the displacement of the neighboring beams \( (\Gamma_2(w_n^2 - w_{n+1}^2 + 2w_n(w_{n+1} - w_n))) \), \( \Gamma_2 = \frac{3}{\gamma} \), so for a single beam or when the displacement of the neighboring beams is equal, this term cancels out as well. The linear and cubic coefficients do not cancel out and they are \( (\Gamma_1 = \frac{2}{\gamma}, \Gamma_3 = \frac{4}{\gamma}) \).

### 2.2 Modal Dynamical System

We reduce the IBVP to a single mode dynamical system via an assumed single mode Galerkin assumption \( (w_n(s, \tau) = q_{n1}(\tau)\phi_1(s)) \) using a harmonic string assumed mode \( (\phi_1(s) = \sqrt{2} \sin(\pi s)) \):

\[ I_1 q_{n\tau\tau} - I_4 q_n[1 + I_5(\frac{1}{2} \Delta q_n^2 + \delta q_n q_{n\tau})] + I_6(\xi q_n + \beta q_{n\tau}) = \eta(1 + \cos(2\Omega\tau)) \sum_{i=0}^{3} \Gamma_i[q_n - q_{n-1}]^i - (-1)^i \Gamma_{n+1}(q_n - q_{n+1}]^i, \]  

(2.2.15)

where \( q_n = q_{n1} \) and the integral coefficients are:

\[ I_0 = \int_0^1 \phi_1 ds = \frac{2\sqrt{2}}{\pi}, \quad I_1 = \int_0^1 \phi_1^2 ds = 1, \quad I_2 = \int_0^1 \phi_1^3 ds = \frac{8\sqrt{2}}{3\pi}, \]  

(2.2.16)

\[ I_3 = \int_0^1 \phi_1^4 ds = \frac{3}{2}, \quad I_4 = \int_0^1 \phi_1 \phi_{1ss} ds = -\pi^2, \]

\[ I_5 = \int_0^1 \phi_1^2 ds = \pi^2, \quad I_6 = \int_0^1 \phi_1 \phi_{1ssss} ds = \pi^4. \]

It is convenient to rescale the maximal response by the average gap \( (x_n = \frac{2n\tilde{\phi}}{\gamma}) \), where \( \tilde{\phi} = \phi_1(\frac{1}{2}) = \sqrt{2} \) and \( \gamma = \frac{\tilde{\phi}}{\tilde{\xi}} \), \( G = G_n + G_{n+1} \) and to rescale time by the unperturbed \( (\eta = 0) \) natural frequency \( (\tilde{\omega}_1 = \sqrt{\tilde{\xi}I_6 - I_4} = \pi \sqrt{1 + \tilde{\xi}^2}) \). The resulting dynamical system for an N beam array (see Figure 2.1), where \( x_0 = x_{N+1} = 0 \), is:
\[ \ddot{x}_n + \beta \dot{x}_n + x_n + \delta x_n^2 \dot{x}_n + \alpha x_3^3 = \]
\[ \eta(1 + \cos(2\Omega \tau)) \sum_{i=0}^{3} \hat{\eta}_{in}(x_n - x_{n-1})^i + \hat{\eta}_{in+1}(x_n - x_{n+1})^i, \]

where the system parameters include a hardening nonlinear stiffness \((\alpha)\), linear damping \((\beta)\), and nonlinear damping\((\delta)\) coefficients:

\[ \alpha = \frac{\pi^2 \tilde{\alpha} \gamma^2}{4(1 + \tilde{\xi} \pi^2)} , \quad \beta = \frac{\pi^3 \tilde{\beta}}{\sqrt{1 + \tilde{\xi} \pi^2}} , \quad \delta = \frac{\pi^3 \tilde{\delta} \gamma^2}{2\sqrt{1 + \tilde{\xi} \pi^2}} , \quad \gamma = \frac{\tilde{G}}{L} \]  

and the forcing parameters include:

\[ \hat{\eta}_{0n} = \frac{\Gamma_0 I_0 \tilde{\phi}}{\tilde{\omega}_1^2 \gamma} , \quad \hat{\eta}_{0n+1} = -\frac{\Gamma_{0n+1} I_0 \tilde{\phi}}{\tilde{\omega}_1^2 \gamma} , \quad \hat{\eta}_{1n} = \frac{\Gamma_{1n}}{\tilde{\omega}_1^2} , \quad \hat{\eta}_{1n+1} = \frac{\Gamma_{1n+1}}{\tilde{\omega}_1^2} , \]

\[ \hat{\eta}_{2n} = \frac{\Gamma_2 I_2 \gamma}{\tilde{\omega}_1^2 \phi} , \quad \hat{\eta}_{2n+1} = -\frac{\Gamma_{2n+1} I_2 \gamma}{\tilde{\omega}_1^2 \phi} , \quad \hat{\eta}_{3n} = \frac{\Gamma_{3n} I_3 \gamma^2}{\tilde{\omega}_1^2 \phi^2} , \]

\[ \hat{\eta}_{3n+1} = \frac{\Gamma_{3n+1} I_3 \gamma^2}{\tilde{\omega}_1^2 \phi^2} , \quad \Omega = \frac{\hat{\Omega}}{\tilde{\omega}_1} ; \quad \tilde{\omega}_1^2 = \pi^2(1 + \tilde{\xi} \pi^2). \]
For a symmetric configuration this equation simplifies to:

\[
\ddot{x}_n + \beta \dot{x}_n + x_n + \delta x_n^2 \dot{x}_n + \alpha x_n^3 = \eta(1 + \cos(2\Omega \tau)) \sum_{i=0}^{3} \hat{\eta}_i [(x_n - x_{n-1})^i + (-1)^{i+1}(x_n - x_{n+1})^i],
\]

(2.2.20)

where \( \hat{\eta}_i = \frac{\Gamma_i I_\gamma^i - 1}{\omega_1^2 \gamma^i - 1} \).

Recall that a symmetric configuration yields negligible bias (\( \hat{\eta}_0 + \hat{\eta}_{\alpha+1} = 0 \)) and, in specific cases (when \( x_{n+1} = x_{n-1} \)), quadratic (\( \hat{\eta}_{2n}(x_n - x_{n-1})^2 + \hat{\eta}_{2n+1}(x_n - x_{n+1})^2 = \frac{\Gamma_2 I_\gamma^2}{\omega_1^2 \phi}(x_{n-1} - x_{n+1} + 2x_n(x_{n+1} - x_{n-1})) \)) forcing. However, an asymmetric configuration exhibits a softening quadratic component in the same cases that the symmetric configuration lacks quadratic forcing (when \( \hat{\eta}_{2n} + \hat{\eta}_{2n+1}(x_n - x_{n+1})^2 = \frac{(\Gamma_2 - \Gamma_{2n+1}) I_\gamma}{\omega_1^2 \phi}(x_n - x_{n+1})^2 > 0 \)) or a hardening quadratic component (when \( \Gamma_2 < \Gamma_{2n+1} \)). Furthermore, note that the system exhibits a decrease in the non-dimensional natural frequency \( (\omega_1 = \sqrt{1 - 4\nu}) \) for both symmetric and asymmetric configurations. Moreover, the system can exhibit a possible elimination of the cubic hardening behavior \( (\alpha = 4\nu \frac{\hat{\eta}_3}{m}) \) or even a softening cubic behavior \( (\alpha < 4\nu \frac{\hat{\eta}_3}{m}) \). Finally, note that the magnitude of the forcing also influences the linear damping \( (Q^{-1} = \frac{\beta}{\omega_1} = \frac{\beta}{\sqrt{1 - 4\nu}}) \).

Substituting \( \nu = \frac{\eta}{\gamma^2 \omega_1^2} \) into Eq. (2.2.20) yields:

\[
\ddot{x}_n + \beta \dot{x}_n + x_n + \delta x_n^2 \dot{x}_n + \alpha x_n^3 = 2\nu(1 + \cos(2\Omega \tau)) \sum_{i=0}^{3} \int_0^{(i+1)\pi} \sin^{i+1}\pi ds \sum [(x_n - x_{n-1})^i + (-1)^{i+1}(x_n - x_{n+1})^i] = 2\nu(1 + \cos(2\Omega \tau))(2x_n - x_{n-1} - x_{n+1} + \frac{1}{m}(x_{n-1}^2 - x_n + 1^2 + 2x_n(x_{n+1} - x_{n-1}))
\]

\[
+3x_n^3 - \frac{9}{2}x_n(x_{n-1} + x_{n+1}) + \frac{9}{2}x_n(x_{n-1}^2 + x_{n+1}^2) - \frac{3}{2}x_n(x_{n-1}^3 - x_{n+1}^3),
\]

(2.2.21)

Note that primary resonance occurs at \( \Omega \sim \frac{1}{2} \). Note also that this formulation is similar to that proposed by LIFSCHITZ & CROSS [30] for their 3-element lumped mass model. However in this model the system parameters are not arbitrary. They are related to each other via the assumptions noted above and both the linear and cubic damping are functions of the viscoelastic damping parameter \( D \).
2.2.1 Single Beam

For a single beam (see Figure 2.2) Eq. (2.2.21) simplifies to:

\[ \ddot{x} + \beta \dot{x} + x + \delta x^2 \dot{x} + \alpha x^3 = \]
\[ 2\nu(1 + \cos(2\Omega\tau)) \sum_{i=0}^{3} (i+1) \int_{0}^{1} \sin^{i+1}\pi s ds(1 + (1)^{i+1})x^i \]
\[ = 2\nu(1 + \cos(2\Omega\tau))(2x + 3x^3), \quad (2.2.22) \]

2.2.2 Three Beams

For a three beam array (see Figure 2.3) the equations become:

\[ \ddot{x}_1 + \beta \dot{x}_1 + x_1 + \delta x_1^2 \dot{x}_1 + \alpha x_1^3 = \]
\[ 2\nu(1 + \cos 2\Omega\tau) \sum_{i=0}^{3} (i+1) \int_{0}^{1} \sin^{i+1}\pi s ds(x_1^i + (-1)^{i+1}(x_1 - x_2)^i) \]
\[ = 2\nu(1 + \cos 2\Omega\tau)(2x_1 - x_2 - \frac{4}{\pi}(x_2^2 - 2x_1x_2) + 3x_1^3 - \frac{9}{2}x_1^2x_2 + \frac{9}{2}x_1x_2^2 - \frac{3}{2}x_2^3), \quad (2.2.23) \]
\[ \ddot{x}_2 + \beta \dot{x}_2 + x_2 + \delta x_2^3 \dot{x}_2 + \alpha x_2^3 = \]

\[
2\nu(1 + \cos 2\Omega \tau) \sum_{i=0}^{3} (i+1) \int_0^1 \sin^{i+1} \pi sds((x_2 - x_1)^i + (-1)^{i+1}(x_2 - x_3)^i)
\]

\[
= 2\nu(1 + \cos 2\Omega \tau)(2x_2 - x_1 - x_3 + \frac{4}{\pi}(x_1^2 - x_3^2 + 2x_2(x_3 - x_1))
\]

\[
+3x_2^3 - \frac{9}{2}x_2^2(x_1 + x_3) + \frac{9}{2}x_2(x_1^2 + x_3^2) - \frac{3}{2}x_1^3 - \frac{3}{2}x_3^3 \tag{2.2.24}
\]

\[ \ddot{x}_3 + \beta \dot{x}_3 + x_3 + \delta x_3^3 \dot{x}_3 + \alpha x_3^3 = \]

\[
2\nu(1 + \cos 2\Omega \tau) \sum_{i=0}^{3} (i+1) \int_0^1 \sin^{i+1} \pi sds((x_3 - x_2)^i + (-1)^{i+1}x_3^i)
\]

\[
= 2\nu(1 + \cos 2\Omega \tau)(2x_3 - x_2 + \frac{4}{\pi}(x_2^2 - 2x_3x_2))
\]

\[
+3x_3^3 - \frac{9}{2}x_3^2x_2 + \frac{9}{2}x_3x_2^2 - \frac{3}{2}x_2^3 \tag{2.2.25}
\]
Chapter 3

Equilibrium Analysis

In this chapter we investigate the trivial equilibrium of the unforced \((\nu = 0)\) system and investigate the modified stability of the three element system augmented by the bias induced by the parametric excitation \((\nu > 0)\).

3.1 Fixed points

3.1.1 Single Beam

We consider the unforced system of Eq. (2.2.22) \((\nu = 0)\) and set the derivatives equal to zero to obtain the equation:

\[
x + \alpha x^3 = x(1 + \alpha x^2) = 0
\]  

This gives us the trivial equilibrium point:

\[
x = 0
\]  

This point is unique since \(\alpha > 0\). When the forcing bias is taken into account, two more equilibrium points are obtained. In this case the equilibrium equation is:

\[
(1 - 4\nu)x + (\alpha - 6\nu)x^3 = x(1 - 4\nu + (\alpha - 6\nu)x^2) = 0
\]  

This gives us the second and third physical equilibrium points equal to:

\[
x = \pm \sqrt{\frac{4\nu - 1}{\alpha - 6\nu}}; \quad \frac{1}{4} < \nu < \frac{\alpha}{6} OR \frac{\alpha}{6} < \nu < \frac{1}{4}
\]  

(3.1.4)
The Jacobian of the unforced system at the point $x = \dot{x} = 0$ is:

$$J = \begin{bmatrix} 0 & 1 \\ -1 & -\beta \end{bmatrix}$$ \hspace{1cm} (3.1.5)

The characteristic equation is:

$$\lambda^2 + c_1 \lambda + c_2 = 0$$ \hspace{1cm} (3.1.6)

Since $c_1 = -tr[J] = \beta > 0$ and $c_2 = det[J] = 1 > 0$, the trivial equilibrium point of the unforced system ($\nu = 0$) is asymptotically stable [41, 34]. The specific types of stability are discussed in Appendix A.

The Jacobian of the single beam system taking into account forcing bias and setting $\dot{x} = 0$ is:

$$J = \begin{bmatrix} 0 & 1 \\ -1 + 4\nu + 3(6\nu - \alpha)x_{eq}^2 & -\beta - \delta x_{eq}^2 \end{bmatrix}$$ \hspace{1cm} (3.1.7)

The Jacobian of the system taking into account the forcing bias at the point $x = \dot{x} = 0$ is:

$$J = \begin{bmatrix} 0 & 1 \\ -1 + 4\nu & -\beta \end{bmatrix}$$ \hspace{1cm} (3.1.8)

The Jacobian of the system taking into account the forcing bias at the points $\dot{x} = 0$, $x = \pm \sqrt{\frac{4\nu - 1}{\alpha - 6\nu}}$ is:

$$J = \begin{bmatrix} 0 & 1 \\ -2(-1 + 4\nu) & -\beta - \delta \frac{4\nu - 1}{\alpha - 6\nu} \end{bmatrix}$$ \hspace{1cm} (3.1.9)

The characteristic equation of the point $(x, \dot{x}) = (0, 0)$ with $\nu > 0$ is:

$$\lambda^2 + \beta \lambda + 1 - 4\nu$$ \hspace{1cm} (3.1.10)

Since $\beta > 0$, the point is asymptotically stable as long as $\nu < \frac{1}{4}$. A detailed analysis of this equilibrium point can be found in Appendix A. Figure 3.1 shows the stability plot of $x$ as a function of $\nu$ for $\alpha = 0.1$. Note that $\nu = \frac{1}{4}$ is a pitchfork bifurcation. This result corresponds to $V_{AC} = 411.8$ $V$ for the parallel plate model.

The characteristic equation of the points $(x, \dot{x}) = (\pm \sqrt{\frac{4\nu - 1}{\alpha - 6\nu}}, 0)$ with $\nu > 0$ is:

$$\lambda^2 + c_3 \lambda + c_4 = 0$$ \hspace{1cm} (3.1.11)
where \( c_3 = -tr[J|_{\pm \sqrt{\frac{4\nu-1}{\alpha-6\nu}}} = \beta + \frac{\delta(4\nu-1)}{\alpha-6\nu} \) and \( c_4 = \det [J|_{\pm \sqrt{\frac{4\nu-1}{\alpha-6\nu}}} = 2(-1 + 4\nu) \). The system is asymptotically stable at these equilibrium points if both \( c_3 \) and \( c_4 \) are positive [34, 41]. Therefore, if \( \nu > \frac{1}{4} \) and either \( \delta > \frac{3}{2} \beta \) and \( \nu > \frac{\delta-\beta}{4\delta-63} \) or \( \delta < \frac{3}{2} \beta \) and \( \nu < \frac{\delta-\beta}{4\delta-63} \) the single beam system is asymptotically stable at the equilibrium points \( x = \pm \sqrt{\frac{4\nu-1}{\alpha-6\nu}} \).

### 3.1.2 Three Beams

We consider the unforced system of Eqs. (2.2.23)-(2.2.25) and set the derivatives equal to zero to obtain the equations:

\[
\begin{align*}
x_1 + \alpha x_1^3 &= x_1(1 + \alpha x_1^2) = 0 \quad (3.1.12) \\
x_2 + \alpha x_2^3 &= x_2(1 + \alpha x_2^2) = 0 \quad (3.1.13) \\
x_3 + \alpha x_3^3 &= x_3(1 + \alpha x_3^2) = 0
\end{align*}
\]

Each beam has the equilibrium value:

\( x_n = 0 \) \quad (3.1.14)
Just as with the single beam, without forcing ($\nu = 0$) the beams are decoupled, and the trivial equilibrium points are asymptotically stable. The decay rates (e.g. spiral or sink) depend on the value of $\beta$ as can be seen in Appendix A).

For non-negligible forcing ($\nu > 0$), we obtain the following:

$$
\dot{x}_1 = y_1 = 0 \\
y_1 = -\beta y_1 - x_1 - \delta x_1^2 y_1 - \alpha x_1^3 + 2\nu(2x_1 - x_2 + \frac{4}{\pi}(2x_1x_2 - x_2^2) + 3x_1^3 + \frac{9}{2}x_1x_2^2 - \frac{9}{2}x_1^2x_2 - \frac{3}{2}x_2^3) = -x_1 - \alpha x_1^3 + 2\nu(2x_1 - x_2 + \frac{4}{\pi}(2x_1x_2 - x_2^2) + 3x_1^3 + \frac{9}{2}x_1x_2^2 - \frac{9}{2}x_1^2x_2 - \frac{3}{2}x_2^3) = x_1[-1 + 4\nu + (6\nu - \alpha)x_1^2 + \frac{16\nu}{\pi}x_2 + 9\nu x_2^2 - 9\nu x_1x_2] + \nu x_2[-2 - \frac{8}{\pi}x_2 - 3x_2^2] = 0 \quad (3.1.15)
$$

$$
\dot{x}_2 = y_2 = 0 \\
y_2 = -\beta y_2 - x_2 - \delta x_2^2 y_2 - \alpha x_2^3 + 2\nu(2x_2 - x_1 - x_3 + \frac{4}{\pi}(2x_2x_3 - 2x - 2x_1 + x_1^2 - x_3^2)) + 3x_2^3 + \frac{9}{2}x_2x_1^2 + \frac{9}{2}x_2x_3^2 - \frac{9}{2}x_2^2x_1 - \frac{9}{2}x_2^2x_3 - \frac{3}{2}x_1^3 - \frac{3}{2}x_3^3) = -x_2 - \alpha x_2^3 + 2\nu(2x_2 - x_1 - x_3 + \frac{4}{\pi}(2x_2x_3 - 2x - 2x_1 + x_1^2 - x_3^2)) + 3x_2^3 + \frac{9}{2}x_2x_1^2 + \frac{9}{2}x_2x_3^2 - \frac{9}{2}x_2^2x_1 - \frac{9}{2}x_2^2x_3 - \frac{3}{2}x_1^3 - \frac{3}{2}x_3^3) = \nu x_1[-2 + \frac{8}{\pi}x_1 - 3x_1^2] + x_2[4\nu - 1 + (6\nu - \alpha)x_2^2 + \frac{16\nu}{\pi}x_3 - \frac{16\nu}{\pi}x_1 + 9\nu x_1^2 + 9\nu x_2^2 - 9\nu x_1x_2 - 9\nu x_2x_3] + \nu x_3[-2 - \frac{8}{\pi}x_3 - 3x_3^2] = 0 \quad (3.1.16)
$$

$$
\dot{x}_3 = y_3 = 0 \\
y_3 = -\beta y_3 - x_3 - \delta x_3^2 y_3 - \alpha x_3^3 + 2\nu(2x_3 - x_2 + \frac{4}{\pi}(-2x_3x_2 + x_2^2)) + 3x_3^3 + \frac{9}{2}x_3x_2^2 - \frac{9}{2}x_3^2x_2 - \frac{3}{2}x_2^3) = -x_3 - \alpha x_3^3 + 2\nu(2x_3 - x_2 + \frac{4}{\pi}(-2x_3x_2 + x_2^2)) + 3x_3^3 + \frac{9}{2}x_3x_2^2 - \frac{9}{2}x_3^2x_2 - \frac{3}{2}x_2^3) = \nu x_2[-2 + \frac{8}{\pi}x_2 - 3x_2^2] + x_3[4\nu - 1 + (6\nu - \alpha)x_3 - \frac{16\nu}{\pi}x_2 + 9\nu x_2^2 - 9\nu x_2x_3] = 0 \quad (3.1.19)
$$
Solving the second equation of each beam we find the fixed points:

\[ x_1 = x_2 = 0; \quad x_1 = \pm \sqrt{\frac{1 - 4\nu}{6\nu - \alpha}}, \quad x_2 = 0 \tag{3.1.21} \]

\[ x_1 = x_2 = x_3 = 0; \quad x_1 = x_3 = 0, \quad x_2 = \pm \sqrt{\frac{1 - 4\nu}{6\nu - \alpha}}, \tag{3.1.22} \]

\[ x_1 = x_2 = 0; \quad x_1 = \pm \sqrt{\frac{1 - 4\nu}{6\nu - \alpha}}, \quad x_2 = 0 \tag{3.1.23} \]

Consequently, the only physical fixed point of the system is the trivial one.

The Jacobian of this system at the trivial equilibrium point is:

\[
J_6 = \begin{bmatrix}
0 & 1 & 0 & 0 & 0 & 0 \\
-1 + 4\nu & -\beta & -2\nu & 0 & 0 & 0 \\
0 & 0 & 0 & 1 & 0 & 0 \\
-2\nu & 0 & -1 + 4\nu & -\beta & -2\nu & 0 \\
0 & 0 & 0 & 0 & 0 & 1 \\
0 & 0 & -2\nu & 0 & -1 + 4\nu & -\beta
\end{bmatrix} \tag{3.1.24}
\]

The characteristic equation of Eq. (3.1.24) is:

\[
\lambda^6 + 3\beta\lambda^5 + [3 - 12\nu + 3\beta^2]\lambda^4 + [\beta^3 - 24\nu\beta + 6\beta]\lambda^3 + \\
[3 - 12\nu\beta^2 + 3\beta^2 - 24\nu + 40\nu^2]\lambda^2 + \\
[3\beta - 24\nu\beta + 40\nu^2\beta]\lambda + [40\nu^2 - 32\nu^3 + 1 - 12\nu] = 0 \tag{3.1.25}
\]

This gives the following eigenvalues:

\[
\lambda_{1,2} = -\frac{1}{2}\beta \pm \frac{1}{2}\sqrt{\beta^2 - 4 + 16\nu}, \\
\lambda_{3,4} = -\frac{1}{2}\beta \pm \frac{1}{2}\sqrt{\beta^2 + 8\sqrt{2}\nu - 4 + 16\nu}, \\
\lambda_{5,6} = -\frac{1}{2}\beta \pm \frac{1}{2}\sqrt{\beta^2 - 8\sqrt{2}\nu - 4 + 16\nu} \tag{3.1.26}
\]

Consequently, if \( \beta > \sqrt{4 + (8\sqrt{2} - 16)\nu} \) and \( \nu < \frac{2 - \sqrt{2}}{4} \) the point is a stable sink node. If \( \beta < \sqrt{4 + (8\sqrt{2} - 16)\nu} \) the point is a stable spiral. If \( \nu > \frac{2 - \sqrt{2}}{4} \) the point is a saddle point. Note that when the \( \lambda^0 \) term is equal to zero, there is a zero eigenvalue, and this is a point
where stability switches over. In order for this term to be positive it is necessary that 
\( \nu \leq \frac{1}{2} - \frac{\sqrt{2}}{4} \) or \( \frac{1}{4} \leq \nu \leq \frac{1}{2} + \frac{\sqrt{2}}{4} \). However, the second condition gives unstable solutions. 
Note that while the single beam system has a lower limit for the forcing term in order for 
its nontrivial equilibrium points to be stable, the three-beam system has an upper limit.

### 3.2 Natural Frequencies

For the unforced system, all three natural frequencies are equal to one (\( \omega_{ni} = 1 \)). Therefore, an internal 1 : 1 : 1 exists and can be excited externally when \( \Omega = \frac{1}{2} \) or parametrically at \( \Omega = 1 \). For the system that includes the forcing bias, as seen in Eq.s (3.1.15)-(3.1.2), the natural frequencies are found by solving for:

\[
K - \omega_n^2 M = 0, \quad (3.2.27)
\]

where:

\[
K = \begin{bmatrix}
1 - 4\nu & 2\nu & 0 \\
2\nu & 1 - 4\nu & 2\nu \\
0 & 2\nu & 1 - 4\nu \\
\end{bmatrix} \quad (3.2.28)
\]

and:

\[
M = \begin{bmatrix}
1 & 0 & 0 \\
0 & 1 & 0 \\
0 & 0 & 1 \\
\end{bmatrix} \quad (3.2.29)
\]

Or:

\[
\begin{bmatrix}
1 - 4\nu - \omega_n^2 & 2\nu & 0 \\
2\nu & 1 - 4\nu - \omega_n^2 & 2\nu \\
0 & 2\nu & 1 - 4\nu - \omega_n^2 \\
\end{bmatrix} = 0 \quad (3.2.30)
\]

Taking the determinant of the matrix on the left side of Eq. (3.2.30) and setting it equal to zero we find that:

\[
0 = (1 - 4\nu - \omega_n^2)((1 - 4\nu - \omega_n^2)^2 - 4\nu^2) - 2\nu[2\nu(1 - 4\nu - \omega_n^2)] = \\
(1 - 4\nu - \omega_n^2)[\omega_n^4 + (8\nu - 2)\omega_n^2 + 1 - 8\nu + 12\nu^2] - 4\nu^2 + 16\nu^3 + 4\nu^2 \omega_n^2 \\
= -\omega_n^6 + (3 - 12\nu)\omega_n^4 + (24\nu - 3 - 44\nu^2)\omega_n^2 + 1 - 12\nu + 44\nu^2 - 48\nu^3 \quad (3.2.31)
\]

Which we solve for to find that:

\[
\omega_{n1} = \sqrt{1 - 6\nu}, \ \omega_{n2} = \sqrt{1 - 4\nu}, \ \omega_{n3} = \sqrt{1 - 2\nu} \quad (3.2.32)
\]
Note that the smaller the value of $\nu$, the closer the three natural frequencies are to each other. So for small $\nu$ internal 1 : 1 : 1 resonance exists. Figure 3.2 shows the values of $\nu$ for which 1 : 1 : 1 internal resonance can be assumed. In the shaded region $0.9515 \leq \omega_{n1} \leq 1$ and $0 \leq \nu \leq 0.0080833$. Hence, in this model the 1 : 1 : 1 internal resonance is anticipated for $0 \leq V_{AC} \leq 74.04 \text{ V}$. A closeup of the shaded area of Figure 3.2 is shown in Figure 3.3.

Figure 3.2: Natural Frequencies, $\omega_{ni}$, as a Function of $\nu$. Internal 1 : 1 : 1 Resonance can be Assumed within Shaded Area.

Figure 3.3: Natural Frequencies, $\omega_{ni}$, as a Function of $\nu$ in the Region Where There is Internal Resonance. (Blowup of the Shaded Region of Figure 3.2.)
Chapter 4

Weakly Nonlinear Asymptotic Analysis

In this chapter we employ an asymptotic multiple-scales method to the weakly nonlinear modal dynamical system derived for both a single and three element array.

4.1 Principle Parametric Resonance of a Single Beam

We apply the Multiple Scales method [41] to Eq. (2.2.22) and represent the beam displacement as:

\[ x(\varepsilon, \tau) = \sum_{i=1}^{3} \varepsilon^i x_i(T_0, T_1, T_2, \ldots) + O(\varepsilon^4), \]  

where \( T_i = \varepsilon^i \tau \) for \( i = 0, 1, 2, \ldots \). The derivatives with respect to \( \tau \) are \( d / d\tau = \frac{dT_0}{d\tau} \frac{\partial}{\partial T_0} + \frac{dT_1}{d\tau} \frac{\partial}{\partial T_1} + \cdots + D_0 + \varepsilon D_1 + \varepsilon^2 D_2 + \cdots \) and \( \frac{d^2}{d\tau^2} = [\frac{d}{d\tau}]^2 = (D_0 + \varepsilon D_1 + \varepsilon^2 D_2 + \cdots)^2 = D_0^2 + 2\varepsilon D_0 D_1 + \varepsilon^2 (D_1^2 + 2D_0 D_2) + \cdots \), where \( D_i = \frac{\partial}{\partial T_i} \). The linear damping coefficient, \( \beta \), and the AC-voltage parameter, \( \nu \), are scaled as \( \beta = \varepsilon^2 \hat{\beta} \) and \( \nu = \varepsilon^2 \hat{\nu} \).

Appendix B shows the equation that results from substituting these values into the beam equation. Expanding this equation out, and breaking into orders of \( \varepsilon \) results in the following equations:

\[ O(\varepsilon) : \quad D_0^2 x_1 + x_1 = 0 \]  

\[ O(\varepsilon^2) : \quad D_0^2 x_2 + x_2 = -2D_0 D_1 x_1 \]
\[ O(\varepsilon^3) : \quad D_0^2 x_3 + x_3 = -D_0 D_1 x_2 - (D_1^2 + 2D_0 D_2)x_1 - \dot{\beta} D_0 x_1 \quad (4.1.4) \]
\[-\delta x_1^2 D_0 x_1 - \alpha x_1^3 + 2\dot{\nu}[2 + \exp(2i\Omega T_0) + \exp(-2i\Omega T_0)]x_1 \]

### 4.1.1 Solutions

The solution of the homogeneous Eq. (4.1.2) is of the form:
\[ x_1 = A(T_1, T_2) \exp(i T_0) + \bar{A}(T_1, T_2) \exp(-i T_0) \quad (4.1.5) \]

Substituting Eq. (4.1.5) into Eq. (4.1.3) results in:
\[ D_0^2 x_2 + x_2 = 2i[-D_1 A \exp(i T_0) + D_1 \bar{A} \exp(-i T_0)] \quad (4.1.6) \]

Setting the secular terms in Eq. (4.1.6) equal to zero yields \( D_1 A = D_1 \bar{A} = 0 \), which requires that both \( A \) and \( \bar{A} \) be independent of \( T_1 \), and therefore \( A = A(T_2) \) and \( \bar{A} = \bar{A}(T_2) \). Setting the right hand side of Eq. (4.1.6) equal to zero and solving for a particular solution of \( x_1 \), we find that \( x_2 = 0 \). Appendix B shows the results of substituting this and Eq. (4.1.5), as well as the detuning relationship, \( \varepsilon^2 \sigma = \Omega - 1 \), into Eq. (4.1.4).

Setting the secular terms of the right hand side of Eq. (B-4) from Appendix B results in the following slowly varying complex evolution equation:
\[-2i D_2 A - i\dot{\beta} A - \delta A^2 \bar{A} - 3aA^2 \bar{A} + 2\dot{\nu}[2A + A \exp(2i\sigma T_2)] = 0, \quad (4.1.7)\]
where \( D_2 = \frac{\partial}{\partial T_2} \) denotes slow time variability. The complex conjugate equation appears in Appendix B and is identical to Eq. (4.1.7).

Substituting the polar ansatz \( A = \frac{a}{2} \exp(i\theta) \) and \( \bar{A} = \frac{a}{2} \exp(-i\theta) \) into Eq. (4.1.7), separating imaginary and real terms, and defining \( \phi = \sigma T_2 - \theta \) yields the following set of autonomous slowly varying evolution equations:
\[ a' = \left[-\frac{\dot{\beta}}{2} + \dot{\nu} \sin(2\phi)\right]a - \frac{\delta}{8}a^3 \quad (4.1.8) \]
\[ a\phi' = (\sigma + \dot{\nu}[2 + \cos(2\phi)])a - \frac{3}{8}\alpha a^3 \quad (4.1.9) \]
where \( ' = D_2 \).

In the steady state no changes with respect to slow time (\( T_2 \)) are observed, and therefore \( a' = \phi' = 0 \). One solution of Eqs. (4.1.8) and (4.1.9) is the trivial one, \( a = 0 \). To solve for the nontrivial solution of these Eqs., the harmonic terms of each of the Eqs. will be isolated, squared, and summed. This process results in the following equation:
\[ \left[\frac{9}{64}\alpha^2 + \frac{\delta^2}{64}\right]a^4 + \left[-\frac{3}{4}\alpha\sigma - \frac{3\dot{\nu}}{2}\alpha + \frac{\delta\dot{\beta}}{8}\right]a^2 + \sigma^2 + 3\nu^2 + \frac{\dot{\beta}^2}{4} + 4\sigma\dot{\nu} = 0 \quad (4.1.10) \]
Multiplying (4.1.10) by \(\varepsilon^4\) gives us:

\[
\left[\frac{9}{64}\alpha^2 + \frac{\delta^2}{64}\right](\varepsilon a)^4 + \left[-\frac{3}{4}\alpha\varepsilon^2\sigma - \frac{3\nu}{2}\alpha + \frac{\delta\beta}{8}\right](\varepsilon a)^2 + (\varepsilon^2\sigma)^2 + 3\nu^2 + \frac{\beta^2}{4} + 4\varepsilon^2\sigma\nu = 0
\]

Substituting the detuning equation into Eq. (4.1.11) gives us:

\[
\left[\frac{9}{64}\alpha^2 + \frac{\delta^2}{64}\right](\varepsilon a)^4 + \left[-\frac{3}{4}\alpha(\Omega - 1) - \frac{3\nu}{2}\alpha + \frac{\delta\beta}{8}\right](\varepsilon a)^2 + (\Omega - 1)^2 + 3\nu^2 + \frac{\beta^2}{4} + 4(\Omega - 1)\nu = 0
\]

The solution of Eq. (4.1.12) for \((\varepsilon a)^2\), the amplitude of the beam displacement, \(x\), is:

\[
(\varepsilon a)^2 = \frac{3\alpha(\Omega - 1) + 6\nu\alpha - \frac{\delta\beta}{2}}{\frac{9}{8}\alpha^2 + \frac{\delta^2}{8}} \pm \sqrt{\left[\frac{\delta\beta}{8} - \frac{3}{4}\alpha(\Omega - 1) - \frac{3\nu}{2}\alpha\right] - \left[\frac{9}{16}\alpha^2 + \frac{\delta^2}{16}\right]\left[(\Omega - 1)^2 + 3\nu^2 + \frac{\beta^2}{4} + 4(\Omega - 1)\nu\right]} - \frac{c_1}{c_6}
\]

where \(c_1 = 24\alpha\), \(c_2 = 24\alpha(2\nu - 1) - 4\delta\beta\), \(c_3 = -\delta^2\), \(c_4 = -2\delta^2(2\nu - 1) - 3\alpha\delta\beta\), \(c_5 = 3(3\alpha^2 - \delta^2)\nu^2 - 3\delta\beta\alpha(2\nu - 1) + \delta^2(4\nu - 1) - \frac{9}{4}\beta^2\alpha^2\), and \(c_6 = 9\alpha^2 + \delta^2\).

Note that all AC voltages which we present in this research are based off of the parallel plate model with a scaled gap of \(9.646 \times 10^{-3}\). Figure 4.1 shows the frequency response curve obtained using Eq. (4.1.14) with the parameter values \(\alpha = 4\), \(\delta = 2\), \(\beta = 0.002\) (\(Q = 500\)), \(\nu = 0.002\) (\(V_{AC} = 36.83V\)). The solid line is stable, the dashed line is unstable. Note that the frequency response plots the response \(\varepsilon a = \frac{\phi_1(s=\frac{1}{\gamma})}{\gamma} q(\tau) = \frac{w(s=\frac{1}{\gamma},\tau)}{\gamma}\), which is the midbeam displacement divided by the scaled gap width. Figure 4.3 shows the frequency response plot for \(\delta = 0\), and the rest of the parameter values identical to those of Figure 4.1. Figure 4.4 shows the frequency response plot for \(\delta = 0.14\), \(\alpha = 0.186\), and the rest of the parameter values identical to those of the previous figures. Figure 4.5 shows the system consisting of the same parameters as Figure 4.4, with \(\delta = 0\). Figure 4.6 shows the frequency response plot for \(\nu = 0.001125\) (\(V_{AC} = 27.62V\)) and the rest of the
Figure 4.1: Frequency Response of A Single Beam System; \( \delta = 2, \alpha = 4, \beta = 0.002 \) \((Q = 500)\), \( \nu = 0.002 \) \((V_{AC} = 36.83V)\).

Figure 4.2: Phase Plot Of A Single Beam System; \( \delta = 2, \alpha = 4, \beta = 0.002 \) \((Q = 500)\), \( \nu = 0.002 \) \((V_{AC} = 36.83V)\).
Figure 4.3: Frequency Response of A Single Beam System; $\delta = 0$, $\alpha = 4$, $\beta = 0.002$ ($Q = 500$), $\nu = 0.002$ ($V_{AC} = 36.83V$).

Figure 4.4: Frequency Response of A Single Beam System; $\delta = 0.14$, $\alpha = 0.186$, $\beta = 0.002$ ($Q = 500$), $\nu = 0.002$ ($V_{AC} = 36.83V$).
Figure 4.5: Frequency Response of A Single Beam System; $\delta = 0$, $\alpha = 0.186$, $\beta = 0.002$ ($Q = 500$), $\nu = 0.002$ ($V_{AC} = 36.83V$).

Figure 4.6: Frequency Response of A Single Beam System; $\delta = 0.14$, $\alpha = 0.186$, $\beta = 0.002$ ($Q = 500$), $\nu = 0.001125$ ($V_{AC} = 27.62V$).
Figure 4.7: Frequency Response of A Single Beam System; $\delta = 0$, $\alpha = 0.186$, $\beta = 0.002$ ($Q = 500$), $\nu = 0.001125$ ($V_{AC} = 27.62V$).

parameter values identical to those of the previous figure. Figure 4.7 graphs the system consisting of the same parameters as Figure 4.6, and $\delta = 0$.

Solving Eq. (4.1.8) at steady state ($a' = \phi' = 0$) for $\phi$ gives us:

$$\phi = \frac{1}{2} \arcsin \left( \frac{1}{2\nu} \left[ \beta + \frac{\delta}{4} (\varepsilon a)^2 \right] \right)$$

(4.1.15)

Substituting Eq. (4.1.13) into Eq. (4.1.15) gives us:

$$\phi = \frac{1}{2} \arcsin \left( a_1 + a_2 [a_3 \Omega + a_4 \pm \sqrt{a_5 \Omega^2 + a_6 \Omega + a_7}] \right),$$

(4.1.16)

where $a_1 = \frac{\beta}{2\nu}$, $a_2 = \frac{\delta}{2\nu(9\alpha^2 + \delta^2)}$, $a_3 = 6\alpha$, $a_4 = -6\alpha + 12\nu\alpha - \delta\beta$, $a_5 = -4\delta^2$, $a_6 = 8\delta^2 - 16\delta^2 \nu - 12\alpha\beta\delta$, $a_7 = 36\alpha^2 \nu^2 - 24\alpha\beta\delta\nu - 9\alpha^2 \beta^2 - 12\delta^2 \nu^2 + 16\delta^2 \nu + 12\alpha\beta\delta - 4\delta^2$.

Figure 4.2 shows the phase plot as a function of frequency corresponding to the parameters used in Figure 4.1.

The bandwidth of Eq. (4.1.14) can be determined by setting $a = 0$ in Eq. (4.1.10) and solving for $\Omega$. We find this way the points where the trivial and nontrivial solutions of $\varepsilon a$ meet. These $\Omega$ values are equal to:
\[
\Omega_{\varepsilon a=0} = 1 - 2\nu \pm \frac{1}{2} \sqrt{4\nu^2 - \beta^2} \tag{4.1.17}
\]

Subtracting the smaller \(\Omega\) value from the larger one in Eq. (4.1.17) gives us a bandwidth (BW) equal to:

\[
BW = \sqrt{4\nu^2 - \beta^2} \tag{4.1.18}
\]

To determine the \(\Omega\) value at the maximum \(\varepsilon a\) we take the partial derivative of \(\varepsilon a\) with respect to \(\Omega\) and set it equal to 0:

\[
\frac{\partial (\varepsilon a)}{\partial \Omega} = \frac{1}{2\varepsilon a} c_1 + \frac{4(2c_3\Omega + c_4)}{c_6\sqrt{c_3\Omega^2 + c_4\Omega + c_5}} = 0 \tag{4.1.19}
\]

Solving Eq. (4.1.19) for \(\Omega_{\text{amax}}\) gives us:

\[
\Omega_{\text{amax}} = -\frac{c_4}{2c_3} + \frac{c_1\sqrt{c_3^2 - 4c_3c_5}}{2c_3\sqrt{c_1^2 - 64c_3}} \tag{4.1.20}
\]

We substitute the original system parameters of Eq. (2.2.22) into Eq. (4.1.20) to obtain:

\[
\Omega_{\text{amax}} = 1 - 2\nu + \frac{3\alpha}{\delta}(\nu - \frac{\beta}{2}). \tag{4.1.21}
\]

Substituting the \(\Omega\) value of Eq. (4.1.20) back into Eq. (4.1.14) gives us a maximum value for \(\varepsilon a\) of:

\[
(\varepsilon a)_{\text{max}} = \sqrt{\frac{2c_2c_3 - c_1c_4 + (c_1^2 + 64c_3c_5)\sqrt{c_4^2 - 4c_3c_5}}{2c_3c_6}} \sqrt{\frac{1}{c_1^2 - 64c_3}}. \tag{4.1.22}
\]

In terms of the original parameters we have that:

\[
(\varepsilon a)_{\text{max}} = 2\sqrt{\frac{2\nu - \beta}{\delta}}. \tag{4.1.23}
\]

Recall that when substituting Eq. (4.1.21) into Eq. (4.1.14) we must choose the plus sign which represents the stable solution, where the maximum value is found.

Substituting the parameter values from Figure 4.1 into Eq. (4.1.17) gives us \(\Omega_1 = 0.9943\) and \(\Omega_2 = 0.9977\). These points were measured by MATLAB using the ginput command, and identical values were obtained. Substituting these values into Eq. (4.1.20) gives us
$\Omega_{\text{amax}} = 1.0020$. Substituting these parameter values into Eq. (4.1.23) gives us $\varepsilon_{\text{amax}} = 0.06325$. These values match the measurements obtained for these points using the ginput command.

4.1.2 Stability of Solutions

The Jacobian of Eq. set (4.1.8) is:

$$J_{\text{polar}} = \begin{bmatrix} -\frac{\hat{\beta}}{2} + \hat{\nu} \sin (2\phi) - \frac{3}{8} \delta a^2 & 2\hat{\nu} \cos (2\phi) a \\ -\frac{3}{4} \alpha a & -2\hat{\nu} \sin (2\phi) \end{bmatrix}$$  \hspace{1cm} (4.1.24)

From the equilibrium Eqs. we have:

$$-\frac{\hat{\beta}}{2} + \hat{\nu} \sin (2\phi) = \frac{\delta}{8} a^2 \hspace{0.5cm} (\text{or} \hspace{0.1cm} a = 0) \hspace{1cm} (4.1.25)$$

$$\hat{\nu} \cos (2\phi) = \frac{3}{8} \alpha a^2 - 2\hat{\nu} - \sigma \hspace{1cm} (4.1.26)$$

Substituting Eqs. (4.1.25) and (4.1.26) into Jacobian (4.1.24) gives us:

$$J_{\text{polar}} = \begin{bmatrix} -\frac{\delta}{4} a^2 - \frac{3}{4} \alpha a^3 - 4\hat{\nu} a - 2\sigma a \\ -\frac{3}{4} \alpha a - \hat{\beta} - \frac{\delta}{4} a^2 \end{bmatrix}$$  \hspace{1cm} (4.1.27)

The Eigenvalue equation of Jacobian (4.1.27) is:

$$\lambda^2 + \left[\frac{\delta}{2} a^2 + \hat{\beta}\right] \lambda + \left[\left(\delta^2 + 9\alpha^2\right) \frac{a^2}{4} + \delta \hat{\beta} - 6\alpha \sigma - 12\alpha \hat{\nu}\right] \frac{a^2}{4} = 0$$ \hspace{1cm} (4.1.28)

Since $\frac{\delta}{2} a^2 + \hat{\beta}$ will always be positive, in order for the nontrivial solution of Eqs. (4.1.8) and (4.1.9) to be stable, it is necessary that $\left(\delta^2 + 9\alpha^2\right) \frac{a^2}{4} + \delta \hat{\beta} - 6\alpha \sigma - 12\alpha \hat{\nu} > 0$. Multiplying this inequality by $\varepsilon^2$ and substituting in the detuning relation ($\varepsilon^2 \sigma = \Omega - 1$) results in the inequality $\left(\delta^2 + 9\alpha^2\right) \frac{(\varepsilon a)^2}{4} + \delta \hat{\beta} - 6\alpha (\Omega - 1) - 12\alpha \hat{\nu} > 0$. Rearranging this equation, we find that $(\varepsilon a)^2 > \frac{24\alpha (\Omega - 1) + 48\alpha \nu - 4\delta \hat{\beta}}{9\alpha^2 + 6\alpha \sigma + 4\delta \hat{\beta}}$. Looking at Eq. (4.1.13) we see that the right hand side of the previous inequality is the part of $(\varepsilon a)^2$ that is not in the square root. Since a square root is always positive, this means in order for $(\varepsilon a)^2$ to be more than the part of it not in the square root, the square root must be added. Hence, the branch of $(\varepsilon a)^2$ where
the square root is added is the stable branch of the solution, and the branch where it is subtracted is the unstable branch. The saddle point occurs at the point where the square root is equal to zero. At this point:

$$\Omega_{saddle} = -\frac{c_4 - \sqrt{c_4^2 - 4c_3c_5}}{2c_3},$$ \hspace{1cm} (4.1.29)

where $c_3$, $c_4$, and $c_5$ are as defined in Eq. (4.1.14).

In terms of the original parameters, Eq. (4.1.29) simplifies to:

$$\Omega_{saddle} = -(2\nu - 1) - \frac{3\alpha\beta}{2\delta} + \frac{\nu}{\delta}\sqrt{\delta^2 + 9\alpha^2} \hspace{1cm} (4.1.30)$$

Substituting this point back into Eq. (4.1.14) gives us:

$$(\varepsilon a)_{saddle} = 2\sqrt{-\frac{\beta}{\delta} + \frac{6\alpha\nu}{\delta\sqrt{9\alpha^2 + \delta^2}}} \hspace{1cm} (4.1.31)$$

Substituting the parameters of Figure 4.1 into Eqs. (4.1.30) and (4.1.31) gives us $\Omega_{saddle} = 1.0022$ and $(\varepsilon a)_{saddle} = 0.0624$. Once again, these values correspond to the values obtained using the ginput command in MATLAB.

Alternatively, the Cartesian ansatz $A = \frac{1}{2}(p+iq)\exp(i\sigma T_2)$ and $\bar{A} = \frac{1}{2}(p-iq)\exp(-i\sigma T_2)$ can be substituted into Eq. (4.1.7), and the following slowly varying evolution equations are obtained:

$$p' = [\sigma + \hat{\nu}]q - \frac{\hat{\beta}}{2}p - \frac{\delta}{8}(p^3 + pq^2) - \frac{3\alpha}{8}(p^2q + q^3) \hspace{1cm} (4.1.32)$$

$$q' = -[\sigma + 3\hat{\nu}]p - \frac{\hat{\beta}}{2}q - \frac{\delta}{8}(p^2q + q^3) + \frac{3\alpha}{8}(p^3 + pq^2) \hspace{1cm} (4.1.33)$$

Since Eq. (4.1.8) neglects the trivial solution of $a = 0$, in order to analyze the stability of this solution, we must analyze the Cartesian system at the point $p^* = q^* = 0$. The Jacobian of this system (Eqs. (4.1.32) and (4.1.33)) at this point ($p^* = q^* = 0$) is:

$$J_{cartesian} = \begin{bmatrix} -\frac{\hat{\beta}}{2} & \sigma + \hat{\nu} \\ -(\sigma + 3\hat{\nu}) & -\frac{\hat{\beta}}{2} \end{bmatrix} \hspace{1cm} (4.1.34)$$

The Eigenvalue equation of Jacobian (4.1.34) is:
\begin{equation}
\lambda^2 + \hat{\beta} \lambda + \left[ \frac{\hat{\beta}^2}{4} + \sigma^2 + 4\sigma \hat{\nu} + 3\nu^2 \right] = 0
d\tag{4.1.35}
\end{equation}

Since \( \hat{\beta} \) is always positive, in order for the trivial solution \( (\varepsilon a = 0) \) to be stable, we require that \( \frac{\hat{\beta}^2}{4} + \sigma^2 + 4\sigma \hat{\nu} + 3\nu^2 > 0 \). Multiplying this inequality by \( \varepsilon^4 \) gives us the inequality
\( \frac{\hat{\beta}^2}{4} + \varepsilon^2 \sigma^2 + 4\varepsilon^2 \sigma \nu + 3\nu^2 > 0 \), and substituting the detuning relation, \( \varepsilon^2 \sigma = \Omega - 1 \), results in
\( \frac{\beta^2}{4} + (\Omega - 1)^2 + 4\nu(\Omega - 1) + 3\nu^2 > 0 \), which expands out to
\( \Omega^2 + (4\nu - 2)\Omega + 1 - 4\nu + 3\nu^2 + \frac{\beta^2}{4} > 0 \).
Solving this equation for \( \Omega \), we find that in order for the trivial solution to be stable:

\begin{equation}
\Omega < 1 - 2\nu - \sqrt{\nu^2 - \frac{\beta^2}{4}} \quad \text{or} \quad \Omega > 1 - 2\nu + \sqrt{\nu^2 - \frac{\beta^2}{4}}
d\tag{4.1.36}
\end{equation}

For the values in Figure 4.1, these values correspond to \( \Omega < 0.9943 \) or \( \Omega > 0.9977 \). These are the same values calculated in Eq. (4.1.17) as the zero-points of the nontrivial solution of \( \varepsilon a \). This means that the trivial solution is stable before and after the zero-points of the nontrivial solution, but not in between them.

To verify that the Cartesian equation corresponds to the polar equation and that all the tools used to perform calculations result in identical solutions, Figure C.1 is plotted in Appendix C. It shows the MATCONT Cartesian solution, the MATCONT Polar solution, the ode45 Cartesian solution, and the ode45 original Eq. (2.2.22) solution.

### 4.2 Internal Resonance of a Three Beam Array

#### 4.2.1 Solutions

When the Multiple Scales method is applied to Eqs. (2.2.23)–(2.2.25), the beam displacements are represented as:

\begin{equation}
x_n(\varepsilon, \tau) = \sum_{i=1}^{3} \varepsilon^i x_n(T_0, T_1, T_2, \ldots) + O(\varepsilon^4),
d\tag{4.2.37}
\end{equation}

where time scales, time derivatives, and small parameters are as defined in section 4.1.

The result of substituting these values into the 3 beam equations is shown in Appendix D. Extracting the terms of order \( \varepsilon \) from the Eqs. (D-9) – (D-11) in Appendix D results in the following equation set:

\begin{equation}
D_0^2 x_{j1} + x_{j1} = 0, \quad j = 1, 2, 3
d\tag{4.2.38}
\end{equation}
The solutions of Eqs. (4.2.38) are of the form:

\[ x_{j1} = A_j(T_1, T_2) \exp(iT_0) + \tilde{A}_j(T_1, T_2) \exp(-iT_0), \ j = 1, 2, 3 \]  

(4.2.39)

Extracting the terms of order \( \varepsilon^2 \) from Eqs. (D-9) – (D-11) of Appendix D gives the following equations:

\[ D_0^2 x_{j2} + x_{j2} = -2D_0 D_1 x_{j1}, \ j = 1, 2, 3 \]  

(4.2.40)

Substituting Eqs. (4.2.39) into Eqs. (4.2.40) results in:

\[
\begin{align*}
D_0^2 x_{12} + x_{12} &= -2D_0 D_1[A_1 \exp(iT_0)] - 2D_0 D_1[\tilde{A}_1 \exp(-iT_0)] = \\
&2i(-D_1 A_1 \exp(iT_0) + D_1 \tilde{A}_1 \exp(-iT_0)) \\
D_0^2 x_{22} + x_{22} &= -2D_0 D_1[A_2 \exp(iT_0)] - 2D_0 D_1[\tilde{A}_2 \exp(-iT_0)] = \\
&2i(-D_1 A_2 \exp(iT_0) + D_1 \tilde{A}_2 \exp(-iT_0)) \\
D_0^2 x_{32} + x_{32} &= -2D_0 D_1[A_3 \exp(iT_0)] - 2D_0 D_1[\tilde{A}_3 \exp(-iT_0)] = \\
&2i(-D_1 A_3 \exp(iT_0) + D_1 \tilde{A}_3 \exp(-iT_0)) \\
\end{align*}
\]

(4.2.41)

Setting the secular terms in Eqs. (4.2.41) equal to zero results in \( D_1 A_n = D_1 \tilde{A}_n = 0 \) which means that \( A_n = A_n(T_2) \) and \( \tilde{A}_n = \tilde{A}_n(T_2) \). Equations (4.2.41) then become:

\[
\begin{align*}
D_0^2 x_{12} + x_{12} &= 0 \\
D_0^2 x_{22} + x_{22} &= 0 \\
D_0^2 x_{32} + x_{32} &= 0, \\
\end{align*}
\]

(4.2.42)

which means that \( x_{n2} = 0 \).

Extracting the terms of order \( \varepsilon^3 \) from Eqs. (D-9) – (D-11) of Appendix D gives the following set of slowly varying evolution equations:

\[
\begin{align*}
D_0^2 x_{13} + x_{13} &= -2D_0 D_1 x_{12} - D_1^2 x_{11} - 2D_0 D_2 x_{11} - \beta D_0 x_{11} - \delta x_{11} D_0 x_{11} - \alpha x_{11}^3 + 2\tilde{\nu}[2 + \exp(2\Omega T_0) + \exp(-2\Omega T_0)](2x_{11} - x_{21}) \\
D_0^2 x_{23} + x_{23} &= -2D_0 D_1 x_{22} - D_1^2 x_{21} - 2D_0 D_2 x_{21} - \beta D_0 x_{21} - \delta x_{21} D_0 x_{21} - \alpha x_{21}^3 + 2\tilde{\nu}[2 + \exp(2\Omega T_0) + \exp(-2\Omega T_0)](2x_{21} - x_{11} - x_{31}) \\
D_0^2 x_{33} + x_{33} &= -2D_0 D_1 x_{32} - D_1^2 x_{31} - 2D_0 D_2 x_{31} - \beta D_0 x_{31} - \delta x_{31} D_0 x_{31} - \alpha x_{31}^3 + 2\tilde{\nu}[2 + \exp(2\Omega T_0) + \exp(-2\Omega T_0)](2x_{31} - x_{21}) \\
\end{align*}
\]

(4.2.43)

Appendix D shows the results of substituting the detuning relation \( \varepsilon^2 \sigma = \Omega - 1 \), Eqs. (4.1.5), and \( x_{n2} = 0 \) into Eqs. (4.2.43). Setting the secular terms of the right side of Eqs.
(D-12)–(D-14) of Appendix D equal to zero results in the following equations:

\[ -2iD_2 A_1 - \hat{\beta}iA_1 - \delta iA_1^2 \hat{A}_1 - 3\alpha A_1^2 \hat{A}_1 + \]
\[ \hat{\nu}[4A_1 - 2A_2 + [2\hat{A}_1 - \hat{A}_2] \exp (2i\sigma T_2)] = 0 \]
\[ -2iD_2 A_2 - \hat{\beta}iA_2 - \delta iA_2^2 \hat{A}_2 - 3\alpha A_2^2 \hat{A}_2 + \]
\[ \hat{\nu}(4A_2 - 2A_1 - 2A_3 + [2\hat{A}_2 - \hat{A}_1 - \hat{A}_3] \exp (2i\sigma T_2)) = 0 \]
\[ -2iD_2 A_3 - \hat{\beta}iA_3 - \delta iA_3^2 \hat{A}_3 - 3\alpha A_3^2 \hat{A}_3 + \]
\[ \hat{\nu}[4A_3 - 2A_2 + [2\hat{A}_3 - \hat{A}_2] \exp (2i\sigma T_2)] = 0 \]

\[ (4.2.44) \]

The identical complex conjugate equations appear in Appendix D.

It should be noted, that taking each \(A_i\) equation from Eqs. (4.2.44) and setting \(A_j = 0; j \neq i\) results in the following set of similar equations, which match Eq. (4.1.7) derived for a single beam.

\[ -2iD_2 A_j - \hat{\beta}iA_j - \delta iA_j^2 \hat{A}_j - 3\alpha A_j^2 \hat{A}_j + 2\hat{\nu}[2A_j + \hat{A}_j \exp (2i\sigma T_2)] = 0, \quad j = 1, 2, 3 \quad (4.2.45) \]

Substituting the polar ansatz \(A_j = \frac{a_j}{2} \exp i\theta_j\) and \(\bar{A}_j = \frac{a_j}{2} \exp -i\theta_j\) and the variables which we defined \(\phi_j = \sigma T_2 - \theta_j\) for \(j = 1 \cdots 3\) and the scaled constants \(\hat{\delta} = \frac{\delta}{8}, \hat{\beta}_* = \frac{\hat{\beta}}{2},\) and \(\hat{\alpha} = \frac{3}{8}\alpha\) into Eqs. (4.2.44) gives the following autonomous slowly varying evolution equations:

\[ a_1' = [-\hat{\beta} + \hat{\nu}\sin (2\phi_1)]a_1 - \hat{\delta}a_1^3 \quad (4.2.46) \]
\[ -\hat{\nu}[\sin (\phi_1 - \phi_2) + \frac{\hat{\nu}}{2} \sin (\phi_1 + \phi_2)]a_2 \]
\[ a_1 \phi_1' = [\sigma + \hat{\nu}(2 + \cos (2\phi_1))]a_1 - \hat{\alpha}a_1^3 \quad (4.2.47) \]
\[ -\hat{\nu}[\cos (\phi_1 - \phi_2) + \frac{\hat{\nu}}{2} \cos (\phi_1 + \phi_2)]a_2 \]

\[ a_2' = [-\hat{\beta} + \hat{\nu}\sin (2\phi_2)]a_2 - \hat{\delta}a_2^3 + \]
\[ \hat{\nu}[(\sin (\phi_1 - \phi_2) - \frac{\hat{\nu}}{2} \sin (\phi_1 + \phi_2))a_1 + \]
\[ (- \sin (\phi_3 - \phi_2) + \frac{\hat{\nu}}{2} \sin (\phi_2 + \phi_3))a_3] \]
\[ a_2 \phi_2' = [\sigma + \hat{\nu}(2 + \cos (2\phi_2))]a_2 - \hat{\alpha}a_2^3 \]
\[ -\hat{\nu}[(\cos (\phi_1 - \phi_2) + \frac{\hat{\nu}}{2} \cos (\phi_1 + \phi_2))a_1 + \]
\[ [\cos (\phi_3 - \phi_2) + \frac{\hat{\nu}}{2} \cos (\phi_2 + \phi_3))a_3] \]

\[ (4.2.48) \]
\[ (4.2.49) \]
\[
\begin{align*}
a'_3 &= [-\tilde{\beta} + \hat{\nu} \sin (2\phi_3)]a_3 - \hat{\alpha}a^3_3 \quad (4.2.50) \\
-\hat{\nu}[\sin (\phi_3 - \phi_2) + \frac{1}{2} \sin (\phi_2 + \phi_3)]a_2 \\
a_3\phi'_3 &= [\sigma + \hat{\nu}(2 + \cos (2\phi_3))]a_3 - \hat{\alpha}a^3_3 \\
-\hat{\nu}[\cos (\phi_3 - \phi_2) + \frac{1}{2} \cos (\phi_2 + \phi_3)]a_2
\end{align*}
\]

where \( \prime = D_2 \). It should be noted, that taking all the \( a_i \) and \( \phi_i \) equations from (4.2.46)–(4.2.52) and setting \( a_j = \phi_j = 0; j \neq i \) results in the following set of equations, which are similar to Eq. (4.1.8) for a single beam.

\[
\begin{align*}
a'_j &= [-\tilde{\beta} + \hat{\nu} \sin (2\phi_j)]a_j - \hat{\alpha}a^3_j \\
a_j\phi'_j &= [\sigma + \hat{\nu}(2 + \cos (2\phi_j))]a_j - \hat{\alpha}a^3_j, \quad j = 1, 2, 3 \quad (4.2.52)
\end{align*}
\]

Alternatively, the Cartesian ansatz \( A_j = \frac{1}{2}(p_j + iq_j) \exp (i\sigma T_2) \) and \( \tilde{A}_j = \frac{1}{2}(p_j - iq_j) \exp (-i\sigma T_2) \) can be substituted into Eqs. (4.2.44), and the following slowly varying evolution equations are obtained:

\[
\begin{align*}
p'_1 &= [\sigma + \hat{\nu}]q_1 - \tilde{\beta} * p_1 - \hat{\delta}(p_1^3 + p_1q_1^2) - \hat{\alpha}(p_1^2 q_1 + q_1^3) - \frac{\hat{\nu}}{2}q_2 \quad (4.2.53) \\
q'_1 &= -[\sigma + 3\hat{\nu}]p_1 - \tilde{\beta} * q_1 - \hat{\delta}(p_1^2 q_1 + q_1^3) + \hat{\alpha}(p_1^3 + p_1q_1^2) + \frac{3\hat{\nu}}{2} p_2 \quad (4.2.54)
\end{align*}
\]

\[
\begin{align*}
p'_2 &= [\sigma + \hat{\nu}]q_2 - \tilde{\beta} * p_2 - \hat{\delta}(p_2^3 + p_2q_2^2) - \hat{\alpha}(p_2^2 q_2 + q_2^3) - \frac{\hat{\nu}}{2}(q_1 + q_3) \quad (4.2.55) \\
q'_2 &= -[\sigma + 3\hat{\nu}]p_2 - \tilde{\beta} * q_2 - \hat{\delta}(p_2^2 q_2 + q_2^3) + \hat{\alpha}(p_2^3 + p_2q_2^2) + \frac{3\hat{\nu}}{2}(p_1 + p_3) \quad (4.2.56)
\end{align*}
\]

\[
\begin{align*}
p'_3 &= [\sigma + \hat{\nu}]q_3 - \tilde{\beta} * p_3 - \hat{\delta}(p_3^3 + p_3q_3^2) - \hat{\alpha}(p_3^2 q_3 + q_3^3) - \frac{\hat{\nu}}{2}q_2 \quad (4.2.57) \\
q'_3 &= -[\sigma + 3\hat{\nu}]p_3 - \tilde{\beta} * q_3 - \hat{\delta}(p_3^2 q_3 + q_3^3) + \hat{\alpha}(p_3^3 + p_3q_3^2) + \frac{3\hat{\nu}}{2} p_2 \quad (4.2.58)
\end{align*}
\]

It should be noted, that taking all the \( p_i \) and \( q_i \) equations from Eqs. (4.2.53)–(4.2.58) and setting \( p_j = q_j = 0; j \neq i \) results in the following set of equations, which are similar to Eq. (4.1.32) for a single beam.

\[
p'_j = [\sigma + \hat{\nu}]q_j - \tilde{\beta} * p_j - \hat{\delta}(p_j^3 + p_jq_j^2) - \hat{\alpha}(p_j^2 q_j + q_j^3) \quad (4.2.59)
\]
\[ q_j' = -[\sigma + 3\nu] q_j - \beta q_j - \delta(p_j^2 q_j + q_j^3) + \alpha(p_j^3 + q_j^2) \quad (4.2.60) \]

Multiplying Eqs. (4.2.53)–(4.2.58) by \( \varepsilon^2 \) results in:

\[ \varepsilon^2 p_1' = [\Omega - 1 + \nu] q_1 - \frac{\beta}{2} p_1 - \delta(\varepsilon^2 p_1^3 + \varepsilon^2 p_1 q_1^2) \quad (4.2.61) \]
\[ -\alpha(\varepsilon^2 p_1^2 q_1 + \varepsilon^2 q_1^3) - \frac{\nu}{2} q_2 \]
\[ \varepsilon^2 q_1' = -[\Omega - 1 + 3\nu] q_1 - \frac{\beta}{2} q_1 - \delta(\varepsilon^2 p_1^2 q_1 + \varepsilon^2 q_1^3) + \]
\[ \hat{\alpha}(\varepsilon^2 p_1^3 + \varepsilon^2 p_1 q_1^2) + \frac{3\nu}{2} p_2 \]
\[ \varepsilon^2 p_2' = [\Omega - 1 + \nu] q_2 - \frac{\beta}{2} p_2 - \delta(\varepsilon^2 p_2^3 + \varepsilon^2 p_2 q_2^2) \quad (4.2.63) \]
\[ -\alpha(\varepsilon^2 p_2^2 q_2 + \varepsilon^2 q_2^3) - \frac{\nu}{2}(q_1 + q_3) \]
\[ \varepsilon^2 q_2' = -[\Omega - 1 + 3\nu] p_2 - \frac{\beta}{2} q_2 - \delta(\varepsilon^2 p_2^2 q_2 + \varepsilon^2 q_2^3) + \]
\[ \hat{\alpha}(\varepsilon^2 p_2^3 + \varepsilon^2 p_2 q_2^2) + \frac{3\nu}{2}(p_1 + p_3) \]
\[ \varepsilon^2 p_3' = [\Omega - 1 + \nu] q_3 - \frac{\beta}{2} p_3 - \delta(\varepsilon^2 p_3^3 + \varepsilon^2 p_3 q_3^2) \quad (4.2.65) \]
\[ -\alpha(\varepsilon^2 p_3^2 q_3 + \varepsilon^2 q_3^3) - \frac{\nu}{2} q_2 \]
\[ \varepsilon^2 q_3' = -[\Omega - 1 + 3\nu] p_3 - \frac{\beta}{2} q_3 - \delta(\varepsilon^2 p_3^2 q_3 + \varepsilon^2 q_3^3) + \]
\[ \hat{\alpha}(\varepsilon^2 p_3^3 + \varepsilon^2 p_3 q_3^2) + \frac{3\nu}{2} p_2 \]

The blue dots in Figures 4.8-4.11 show the solution of these equations for different parameter sets obtained by the MATLAB continuation toolbox MATCONT [12]. Note that there is a saddle node bifurcation below the maximum response and that similar to the single beam, the lower branch is unstable. Note also that the stable \( x_1 \) and \( x_3 \) solutions will be shown to be out-of-phase with that of \( x_2 \) in Chapter 5. Solutions of additional parameter sets can be found in Appendix E. It should be noted that in all of these figures \(|\varepsilon a_1| = |\varepsilon a_3|\).

A second type of interaction can be seen in Figures 4.10-4.11. The previous figures of the three beam solutions look the same as the single beam solutions. Here we see a second peak. Note that the lower branches are stable below their respective saddle node bifurcation points. Note also that the stable upper and lower branch solutions will be
Figure 4.8: Frequency Response of First and Third Beams of a 3-Beam Array; $\delta = 2$, $\alpha = 4$, $\beta = 0.002$ ($Q = 500$), $\nu = 0.002$ ($V_{AC} = 36.83V$).

Figure 4.9: Frequency Response of Middle Beam of a 3-Beam Array; $\delta = 2$, $\alpha = 4$, $\beta = 0.002$ ($Q = 500$), $\nu = 0.002$ ($V_{AC} = 36.83V$).
Figure 4.10: Frequency Response of First and Third Beams of a 3-Beam Array; $\delta = 2$, $\alpha = 4$, $\beta = 0.002$ ($Q = 500$), $\nu = 0.0068445$ ($V_{AC} = 68.12V$).

Figure 4.11: Frequency Response of Middle Beam of a 3-Beam Array; $\delta = 2$, $\alpha = 4$, $\beta = 0.002$ ($Q = 500$), $\nu = 0.0068445$ ($V_{AC} = 68.12V$).
Figure 4.12: Frequency Response of First and Third Beams of a 3-Beam Array; $\delta = 0.15$, $\alpha = 4$, $\beta = 0.002$ ($Q = 500$), $\nu = 0.00585$ ($V_{AC} = 69.99V$).

Figure 4.13: Frequency Response of Middle Beam of a 3-Beam Array; $\delta = 0.15$, $\alpha = 4$, $\beta = 0.002$ ($Q = 500$), $\nu = 0.00585$ ($V_{AC} = 10.89V$).
shown in Chapter 5 to be out-of-phase and in-phase respectively. In Appendix F the $p$ and $q$ components of Figures E.16 and E.17 can be seen.

Figures 4.12 and 4.13 show a third type of interaction, which consists of two coexisting stable solutions, one of which includes a quasiperiodic response. Note that we found quasiperiodic responses on the lower stable branch for frequencies between $\Omega = 1.0025$ and $\Omega = 1.0038$.

![Figure 4.14](image)

Figure 4.14: Frequency Response of First and Third Beams of a 3-Beam Array; $\delta = 2$, $\alpha = 4$, $\beta = 0.002$ ($Q = 500$), $\nu = 0.00585$ ($V_{AC} = 69.99V$).

Figures 4.14 and 4.15 plot the same parameters with a nonzero nonlinear damping of $\delta = 2$. Here we find no quasiperiodic solution. Hence, we determine that the quasiperiodicity of the in-plane branch is governed by the nonlinear damping as there appears to be a threshold of $\delta$ between $\delta = 0.015$ and $\delta = 2$ which eliminates the Hopf bifurcations on the upper in-phase branch.

### 4.2.2 Stability of Solutions

Taking the Jacobian of the Cartesian system (Eqs. (4.2.61) – (4.2.65)) at the point $p_1^* = q_1^* = 0$ we find that:
The Eigenvalue equation of Jacobian (4.2.67) is:

$$\lambda^6 + b_1 \lambda^5 + b_2 \lambda^4 + b_3 \lambda^3 + b_4 \lambda^2 + b_5 \lambda + b_6 = 0,$$

(4.2.68)

where $b_1 = 6\beta$,

$b_2 = 12\nu \Omega + 12\nu^2 - 12\nu + 15\beta^2 + 3 + 3\Omega^2 - 6\Omega$,

$b_3 = 12\beta + 12\beta \Omega^2 - 48\nu \beta - 24\beta \Omega + 20\beta^3 + 48\nu \beta \Omega + 48\nu^2 \beta$,

$b_4 = 72\nu^3 \Omega + \frac{117}{4}\nu^4 - 12\nu^3 - 36\beta^2 \Omega + 3 + 64\nu^2 + 24\nu \Omega^3 + 18\Omega^2 - 128\nu^2 \Omega - 72\nu \Omega^2 + 15\beta^4 + 72\nu^2 \beta^2 - 72\nu^3 + 18\beta^2 \Omega^2 - 12\Omega - 24\nu + 64\nu^2 \Omega^2 + 3\Omega^4 + 72\nu \Omega + 72\nu \beta^2 \Omega - 72\nu \beta^2 + 18\beta^2$, 

(4.2.67)
Subtracting the smaller \( \Omega \) value from the larger one in Eq. (4.2.69) gives us a bandwidth solution can exist for this value of \( \beta \) no value of \( \nu \) where two peaks corresponding to coexisting out-of-phase and in-phase solutions. In Figure 4.17, where \( b_6 = 0 \) (b > 0) the other coefficients are positive as well. Setting \( b_6 = 0 \) allows us to study the stability of the zero solution as a function of \( \Omega, \beta, \) and \( \nu \).

Solving the equation \( b_6 = 0 \) yields two real solutions for \( \Omega \) for \( \nu > \beta \):

\[
\Omega_{b_6=0} = 1 - 2\nu \pm \sqrt{\nu^2 - \beta^2}, \quad (4.2.69)
\]

and the fourth order equation:

\[
4\Omega^4 + (32\nu - 16)\Omega^3 + (68\nu^2 - 96\nu + 8\beta^2 + 24)\Omega^2 \\
+ (48\nu^3 + 32\nu\beta^2 - 16\beta^2 + 96\nu - 136\nu^2 - 16)\Omega \\
+ (9\nu^4 - 48\nu^3 + 36\nu^2\beta^2 + 68\nu^2 - 32\nu\beta^2 - 32\nu + 4 + 8\beta^2 + 4\beta^4) = 0 \quad (4.2.70)
\]

Subtracting the smaller \( \Omega \) value from the larger one in Eq. (4.2.69) gives us a bandwidth (BW3) equal to:

\[
BW_3 = 2\sqrt{\nu^2 - \beta^2} \quad (4.2.71)
\]

Figures 4.16-4.19 show one of the parameters \( \beta \) and \( \nu \) as a function of the six roots \( \Omega_{b_6=0} \) for a fixed values of the second parameter. Additional plots of other fixed parameters can be found in Appendix G. In Figure 4.16, where \( \beta = 0.001 \) (Q = 1000), all six roots of \( b_6 = 0 \) have solutions, and for particular values of \( \nu \) there are four simultaneous solutions, which means that the frequency response of the system under these parameters will have two peaks corresponding to coexisting out-of-phase and in-phase solutions. In Figure 4.17, where \( \beta = 0.004 \) (Q = 250), the system has four real roots of the equation \( b_6 = 0 \). For no value of \( \nu \) however do all four of these roots coexist, and therefore only a single peak solution can exist for this value of \( \beta \) corresponding to an out-of-phase solution.
Figure 4.16: $\nu$ as a Function of $\Omega_{b_0} = 0; \beta = 0.001 (Q = 1000)$. 

Figure 4.17: $\nu$ as a Function of $\Omega_{b_0} = 0; \beta = 0.004 (Q = 250)$. 
Figure 4.18: $\beta$ as a Function of $\Omega_{b_0=0}$; $\nu = 0.002$ ($V_{AC} = 36.83V$).

Figure 4.19: $\beta$ as a Function of $\Omega_{b_0=0}$; $\nu = 0.007$ ($V_{AC} = 68.90V$).
In Figure 4.18, where $\nu = 0.002$ ($V_{AC} = 36.83V$), there are four real roots of $b_6 = 0$, two of which are on the verge of disappearing. It appears that for a few values of $\beta$ a two peak solution can be realized, and for a wider range of values of $\beta$, a single peak solution to the system equations exists (as seen in Figures 4.8-4.9 in section 5.1.2) and in Appendix E. In Figure 4.19, where $\nu = 0.007$ ($V_{AC} = 68.90V$), we see that the first five roots of $b_6 = 0$ have real (unique) solutions. It appears that both single peak and two peak solutions (as seen in Figures 4.10-4.11 in section 4.2.1) are possible depending on the value of $\nu$.

The bifurcation diagram of Figures 4.8 and 4.9 can be seen in Figure 4.20. Note that regions $I$ and $IV$ contain a unique stable trivial solution. Region $II$ contains an unstable trivial solution and a stable out of phase solution. Region $III$ contains an unstable trivial solution a stable out of phase solution and an unstable out of phase solution. Note that this bifurcation diagram is identical to that of a single beam.

The bifurcation diagram of Figures 4.10 and 4.11 can be seen in Figure 4.21. Note that regions $I$ and $VII$ contain a unique stable trivial solution. Region $II$ contains a stable out of phase solution and an unstable trivial solution. Regions $III$ and $VI$ contain a stable trivial solution, and a stable and unstable out of phase solution. Region $IV$ contains an unstable trivial solution, an unstable out of phase solution and coexisting stable in phase and out of phase solutions. Region $V$ contains a stable trivial solution and coexisting stable and unstable in phase and out of phase solutions.

Note that the bifurcation structure of Figures 4.12 and 4.13 with $\delta = 0$ includes 7 regions:

i a unique stable zero solution;

ii an unstable zero solution and a stable out-of-phase solution;

iii a stable zero solution, an unstable out-of-phase solution, and a stable out-of-phase solution;

iv an unstable zero solution, a stable in-phase solution, and unstable and stable out-of-phase solutions;

v a stable zero solution, unstable and stable in-phase solutions, and unstable and stable out-of-phase solutions;

vi a stable zero solution, two unstable in-phase solutions (the upper of these between two Hopf bifurcations), and unstable and stable out-of-phase solutions;

vii identical to region v.
Figure 4.20: Bifurcation Diagrams; $\delta = 2$, $\alpha = 4$, $\beta = 0.002$ ($Q = 500$), $\nu = 0.002$ ($V_{AC} = 36.83V$).
Figure 4.21: Bifurcation Diagrams; $\delta = 2$, $\alpha = 4$, $\beta = 0.002$ ($Q = 500$), $\nu = 0.0068445$ ($V_{AC} = 68.12V$).
A nine region bifurcation structure is anticipated for $0.15 < \delta < \delta_{threshold}$. This structure is anticipated to augment Figure 4.21 by dividing its region $IV$ into three regions. Two of which are identical to that of region $V$ and one in-between which includes an unstable in-phase solution between two Hopf bifurcation points.
Chapter 5

Strongly Nonlinear Numerical Analysis

In this chapter we present the strongly nonlinear numerical analysis. We use a 4th order Runge-Kutta solver [42] to perform the numerical integrations. The solver uses standard local error control techniques to monitor the error at each time step. Throughout each time step the solver computes the final state values and calculates the local error. It compares this error with the acceptable error which was defined by the user. If the local error of any state is greater than the acceptable error, the time step is reduced and the solver tries again. The results of this section were derived for relative and absolute tolerances of $10^{-10}$.

In Section 5.1 we compare the numerical integration to the Multiple Scales solutions obtained in Chapter 4. In Section 5.2 we present the steady state time series responses, the phase plane and Poincaré maps, and the power spectra. Closed phase plane orbits represent periodic solutions. Quasiperiodic phase planes contain several non-repeatable orbits. A Poincaré map [40] is defined by stroboscopically sampling the steady state response at a constant period of $\frac{\pi}{\Omega}$, where $\Omega$ is the forcing frequency. Recall that because the forcing term is of the form $\cos^2 \Omega \tau$ the system is in actuality being forced at a frequency of $2\Omega$. Periodic solutions are characterized by maps containing a finite number of points and quasiperiodic solutions are composed of continuous closed orbits. The power spectra were calculated using the MATLAB FFT routine [8]. Power spectra provide information on the frequency content of the solution, and contribute to the analysis of the periodic or quasiperiodic behavior of the response.
5.1 Validation of Asymptotics

In this section we validate the asymptotic response solved for in Chapter 4 for parameter sets describing weakly nonlinear interaction. We perform a numerical integration of the system for a set of parameters and a number of forcing frequencies ($\Omega$). We then calculate the steady state amplitude for each frequency by taking the average of the maximum and minimum values of the steady state solution. We plot these points over the asymptotic frequency response plot for the same set of parameters.

5.1.1 Single Beam

In Figures 4.1, 4.3, 4.4, 4.5, 4.6, 4.7 the o-s represent the numerical verification using Eq. (2.2.22), and the x-es represent the numerical verification based on a parallel plate representation of the electrostatic force. As can be seen from these figures, as the forcing, $\nu$, is increased or the nonlinear stiffness, $\alpha$, or nonlinear damping, $\delta$, is decreased, there is a decrease in accuracy of the multiple scales approximation. In addition to our modal differential equations growing apart from our multiple scales approximation as these parameters are changed, the parallel plate model drifts away from both the approximation and the numerical solution to our equations. Note that due to memory and time constraints, some frequencies did not reach steady state in the ode45 solution with the smaller nonlinear parameters.

5.1.2 Three Beams

In Figures 4.8-4.11 the dots are the MATCONT solution to the Cartesian evolution equations, the o-s are MATLAB’s solution to the Cartesian evolution equations using ode45, and the x-es are MATLAB’s solution to the ordinary differential Eqs. (2.2.23)–(2.2.25) using ode45. We find a very good agreement between the different results. We compute a maximal error (defined by $\left| \frac{|e_a|_{ODE} - |e_a|_{MATcont}}{|e_a|_{ODE}} \right| \times 100$) of 1.92% between the asymptotic and numerical solutions for the parameter set belonging to Figure 4.9 and a frequency of $\Omega = 0.9965$ corresponding to a response of $|e_a| = 0.07$. For Figure 4.11 we compute a maximal error (defined by $\left| \frac{\Omega_{ODE} - \Omega_{MATcont}}{\Omega_{ODE}} \right| \times 100$) of 0.05% between the asymptotic and numerical solutions for a frequency of $\Omega = 0.965$ corresponding to a response of $|e_a| = 0.0225$. Thus, we conclude that the asymptotic solution is valid for the entire parameter range that we analyzed.

It should be noted that as $\nu$ is made smaller, it takes a longer time for MATLAB to
obtain the steady state solution of the ODEs, but if it’s run for a long enough time we find that the solution fits the solution to the polar evolution equations which is obtained within a matter of minutes. Therefore, as the forcing is decreased the usefulness of the Multiple Scales approximation increases. Due to time/memory constraints MATLAB can not be used to solve the ordinary differential equations for all sets of parameters. In sections 6.4.2 and 6.4.3 MATLAB interpreter version of ODE45 could not handle the parameters we extracted from the experiment, and we obtained our numerical verification using a fourth order Runge-Kutta solver in Fortran [42]. The parameters used in each graph are listed under the figure.

Figures 5.1 and 5.2 show an example of a parameter set for which the MATLAB solver did not produce good results.

![Graph](image)

Figure 5.1: Frequency Response of First and Third Beams of a 3-Beam Array; $\delta = 0.14$, $\alpha = 0.186$, $\beta = 0.002$ ($Q = 500$), $\nu = 0.0006$ ($V_{AC} = 20.17V$).

### 5.2 Periodic Response

A periodic response is characterized by a repetitive time series, a closed orbit in the phase plane, a finite number of Poincaré points and by discrete frequency peaks in its power spectrum.
Figure 5.2: Frequency Response of Middle Beam of a 3-Beam Array; $\delta = 0.14$, $\alpha = 0.186$, $\beta = 0.002$ ($Q = 500$), $\nu = 0.0006$ ($V_{AC} = 20.17V$).

### 5.2.1 Single Beam

Figure 5.3 shows the time series, Poincaré map, and power spectrum of the system modeled in Figure 4.1. Figure 5.4 shows the time series, Poincaré map, and power spectrum of the system modeled in Figure 4.3. Figures 5.5 and 5.6 have the same parameters as Figures 5.3 and 5.4 except for the larger forcing amplitude of $\nu = 0.00808$ ($V_{AC} = 74.00V$). Notice that a nonlinear damping term of $\delta = 2$ does not influence the steady state amplitude of the system. Note that the power spectra show a single discrete frequency peak. Also note that both Poincaré maps, as would be expected of a parametrically excited system, have two Poincaré points.

### 5.2.2 Three Beams

The time series of Figures E.8 and E.9 for $\Omega = 0.997$ can be seen in Figure 5.7, the phase plane and Poincaré map of this system at this point can be seen in Figure 5.9, and the power spectrum can be seen in Figure 5.8. Notice from the time series that the first and third beam are in-phase with each other and of equal amplitude, and the middle beam is out-of-phase with the outer beams of of slightly larger amplitude. Note the expected two
Figure 5.3: Time Series, Poincaré Map, and Power Spectrum Of A Single Beam System; $\delta = 2, \alpha = 4, \beta = 0.002 (Q = 500), \nu = 0.002 (V_{AC} = 36.83V)$.

Figure 5.4: Time Series, Poincaré Map, and Power Spectrum Of A Single Beam System; $\delta = 0, \alpha = 4, \beta = 0.002 (Q = 500), \nu = 0.002 (V_{AC} = 36.83V)$.
Figure 5.5: Time Series, Poincaré Map, and Power Spectrum Of A Single Beam System; \( \delta = 2, \alpha = 4, \beta = 0.002 \) \( (Q = 500), \nu = 0.00808 \) \( (V_{AC} = 74.00V) \).

Figure 5.6: Time Series, Poincaré Map, and Power Spectrum Of A Single Beam System; \( \delta = 0, \alpha = 4, \beta = 0.002 \) \( (Q = 500), \nu = 0.00808 \) \( (V_{AC} = 74.00V) \).
Poincaré points and the single discrete frequency peak in the power spectrum.

![Graph](image)

Figure 5.7: Time Series of 3-Beam Array at $\Omega = 0.997; \delta = 0.14, \alpha = 0.186, \beta = 0.002 \ (Q = 500), \nu = 0.001 \ (V_{AC} = 26.04V)$.

The time series for Figures E.16 and E.17 are shown in Figures 5.10 (for $\Omega = 0.98$) and 5.11 (for $\Omega = 0.9966$). Notice that the first set of time series, as with the single peak response consists of the two outer beams being in phase and of equal amplitude, and the middle beam being out of phase with the outer beams and of slightly larger amplitude. In the second time series all three responses are in phase, but the amplitude of the middle beam is significantly larger than that of the outer beams. The power spectra (for the same two points) are shown in Figures 5.12 and 5.13. Note that we now see two pronounced discrete frequency peaks in the power spectrum of Figure 5.12. The phase planes and Poincaré maps for these same two points are shown in Figures 5.14 and 5.15. Again, note the expected two Poincaré points.

### 5.3 Quasiperiodic Response

While the LIFSCHITZ & CROSS [30] lumped-mass array included several coexisting stable and unstable periodic solutions they did not reveal the existence of any quasiperiodic energy transfer between the neighboring beams. Similar coupled Duffing equations have
Figure 5.8: Power Spectra of 3-Beam Array at $\Omega = 0.997; \delta = 0.14, \alpha = 0.186, \beta = 0.002 \ (Q = 500), \nu = 0.001 \ (V_{AC} = 26.04V)$.

Figure 5.9: Poincaré Maps of 3-Beam Array at $\Omega = 0.997; \delta = 0.14, \alpha = 0.186, \beta = 0.002 \ (Q = 500), \nu = 0.001 \ (V_{AC} = 26.04V)$. 
Figure 5.10: Time Series of 3-Beam Array at $\Omega = 0.98; \delta = 2, \alpha = 4, \beta = 0.002 (Q = 500), \nu = 0.0065714 (V_{AC} = 66.76V)$.

Figure 5.11: Time Series of 3-Beam Array at $\Omega = 0.9966; \delta = 2, \alpha = 4, \beta = 0.002 (Q = 500), \nu = 0.0065714 (V_{AC} = 66.76V)$. 
Figure 5.12: Power Spectra of 3-Beam Array at $\Omega = 0.98; \delta = 2, \alpha = 4, \beta = 0.002 (Q = 500), \nu = 0.0065714 (V_{AC} = 66.76V)$.

Figure 5.13: Power Spectra of 3-Beam Array at $\Omega = 0.9966; \delta = 2, \alpha = 4, \beta = 0.002 (Q = 500), \nu = 0.0065714 (V_{AC} = 66.76V)$. 
Figure 5.14: Poincaré Maps of 3-Beam Array at $\Omega = 0.98$; $\delta = 2$, $\alpha = 4$, $\beta = 0.002$ ($Q = 500$), $\nu = 0.0065714$ ($V_{AC} = 66.76V$).

Figure 5.15: Poincaré Maps of 3-Beam Array at $\Omega = 0.9966$; $\delta = 2$, $\alpha = 4$, $\beta = 0.002$ ($Q = 500$), $\nu = 0.0065714$ ($V_{AC} = 66.76V$).
been shown to exhibit numerous examples of both quasiperiodic and chaotic responses in addition to multiple coexisting periodic solutions. These aperiodic solutions have been analytically shown to emerge from various internal resonances and combination resonances \cite{41} as an outcome of a Hopf bifurcation. In their numerical analysis of a three beam array subject to a parallel plate model electrodynamic force GUTSCHMIDT & GOTTLIEB \cite{20} found quasiperiodic solutions. We ran numerical simulations for similar parameters to those that they used on Eqs. (2.2.23)-(2.2.25) with a linearized electrostatic force. Our parameter set consisted of a linear damping term of $\beta = 0.002$ ($Q = 500$), a nonlinear damping term of $\delta = 0$, a nonlinear stiffness term of $\alpha = 4$, a forcing magnitude of $\nu = 0.00585$ ($V_{AC} = 62.99V$) and a forcing frequency of $\Omega = 1.0029$. The time series of the system under these parameters can be seen in Figure 5.16. The first and third slowly varying envelopes appear to be in-phase with each other, and out-of-phase with the second envelope. The envelope amplitude of the responses of the first and third beams is half that of the response of the middle beam. The signals of all three beams have slightly different periods of recurrence. The power spectra of this system are shown in Figure 5.17. Notice that in contrast to the sharp peaks we saw in the power spectra of the periodic responses, here the frequency contents at the input forcing frequency $\Omega$ and $3\Omega$ are wide-banded. We note that this spectral analysis doe not resolve the slowly varying periodicity of the beat ($\approx 657$ nondimensional time units) which correspond to a short frequency of 0.0015 nondimensional frequency units. In Figure 5.18 we show the phase plots and Poincaré maps for the quasiperiodic solution. Notice that the Poincare’ maps form two tori as is expected of a quasiperiodic solution that evolves from a period doubled response near the system principle parametric resonance (e.g. Figure 5.3). Figure 5.19 shows the Poincaré map (without the phase plot). Note that the Poincaré map is composed of 1596 points.

Note that the second coexisting solution consists of out-of-phase equal-amplitude responses. The time series, power spectra, and Poincaré maps of these solutions taken at $\Omega = 0.992$ are shown in Figures 5.20-5.22, respectively. Note the equal amplitudes and that $x_2$ is out of phase with the other two responses in Figure 5.20. Note the discrete frequency peak in Figure 5.21. Finally, notice the two Poincaré points in Figure 5.22.

The time series obtained using the same parameters at the same frequency where the quasiperiodic solution was found ($\Omega = 1.0029$) for our system on the parallel plate model for the electrostatic force can be seen in Figure 5.23. Notice that it resembles that of our model and has a similar maximum beat response. Figure 5.24 shows the Poincaré map obtained from the parallel plate model. Note that the torus has the same form as that of our model. Thus, as both strongly nonlinear parametric excitation of the parallel plate model and the linearized excitation demonstrate similar quasiperiodic behavior for small
Figure 5.16: Time Series of 3-Beam Array at $\Omega = 1.0029; \delta = 0, \alpha = 4, \beta = 0.002 (Q = 500), \nu = 0.00585 (V_{AC} = 10.89V)$.

Figure 5.17: Power Spectra of 3-Beam Array at $\Omega = 1.0029; \delta = 0, \alpha = 4, \beta = 0.002 (Q = 500), \nu = 0.00585 (V_{AC} = 10.89V)$.
Figure 5.18: Phase Planes and Poincaré Maps of 3-Beam Array at $\Omega = 1.0029$; $\delta = 0$, $\alpha = 4$, $\beta = 0.002$ ($Q = 500$), $\nu = 0.00585$ ($V_{AC} = 10.89V$).

Figure 5.19: Poincaré Maps of 3-Beam Array at $\Omega = 1.0029$; $\delta = 0$, $\alpha = 4$, $\beta = 0.002$ ($Q = 500$), $\nu = 0.00585$ ($V_{AC} = 10.89V$).
Figure 5.20: Time Series of 3-Beam Array at $\Omega = 0.992; \delta = 0, \alpha = 4, \beta = 0.002 (Q = 500), \nu = 0.00585 (V_{AC} = 10.89V)$.

Figure 5.21: Power Spectra of 3-Beam Array at $\Omega = 0.992; \delta = 0, \alpha = 4, \beta = 0.002 (Q = 500), \nu = 0.00585 (V_{AC} = 10.89V)$. 
Figure 5.22: Phase Planes and Poincaré Maps of 3-Beam Array at $\Omega = 0.992$; $\delta = 0$, $\alpha = 4$, $\beta = 0.002$ ($Q = 500$), $\nu = 0.00585$ ($V_{AC} = 10.89V$).

Amplitude response (e.g. $x_2 \sim 0.1$), we cannot exclude the existence of additional tori or possible chaotic dynamics for larger amplitude response (e.g. $x_2 \sim 0.3 - 0.5$).
Figure 5.23: Time Series of Parallel Plate Model of 3-Beam Array at $\Omega = 1.0029$; $\delta = 0$, $\alpha = 4$, $\beta = 0.002$ ($Q = 500$), $\nu = 0.00585$ ($V_{AC} = 62.99V$).

Figure 5.24: Poincaré Maps of Parallel Plate Model of 3-Beam Array at $\Omega = 1.0029$; $\delta = 0$, $\alpha = 4$, $\beta = 0.002$ ($Q = 500$), $\nu = 0.00585$ ($V_{AC} = 62.99V$).
Chapter 6

Experimental Analysis

In this chapter we describe the fabrication of the sample and the experimental set up. We present the experimentally obtained results, compare them to the theory in order to estimate parameter values, and discuss our results.

6.1 Design and Manufacturing

6.1.1 Fabrication Process

We used 0.5 mm thick Silicon wafers covered with a 100 nm thick layer of $Si_3N_4$ manufactured by LioniX. Initially we started with a 125 µm membrane. We later switched to 300 µm and 500 µm membranes. The membranes were not exactly the measurements indicated by manufacturer. They mostly measured slightly larger, but sometimes smaller as well. The membrane of the sample which was tested measured 508 x 509 µm. We first broke an individual membrane from the wafer and cleaned it with acetone, methanol, isopropane, and diodized water, after which we blew it dry with nitrogen.

We then heated the sample at 180°C for five minutes to evaporate any water that may have remained on the surface. Next, we applied three layers of PMMA, spinning each layer at 3000 rpm for 1 minute, and then baking it on the hotplate for 5 minutes at 180°C. The first two layers applied were 495A6 PMMA and the third was 950A2. The expected thickness of the PMMA on our samples was 650 nm. Measured values varied between 410 nm and 780 nm. As the photoresist aged the thickness of the layers it formed decreased. On the sample that came out successfully and upon which tests were performed, the PMMA measure 665 nm thick.
After the PMMA was spun, the sample was placed in the Jeol T300 scanning electron microscope (SEM) (see Figure 6.2) under a vacuum of $10^{-5}$ bar and patterned by electrolithography. Figures 6.3 and 6.4 show the final pattern we used. The pattern is symmetric, and consists of an electrode, a single beam, an electrode, and then three beams on each side, with a joint electrode used by both sides positioned in the center of the membrane. The beams were designed to be 0.8 $\mu$m wide, and the electrodes were supposed to be 8 $\mu$m wide. The gap width is designed to be 6 $\mu$m. We wrote on the sample at a magnification of 75x with a current of 200 pA and a dose of $300 \frac{uC}{cm^2}$. The actual measurements turned out to be about 8.5 $\mu$m wide electrodes, about 1.75 $\mu$m wide beams, and gaps of about 4.9 $\mu$m wide. The center to center distance between beams was designed to be 3.4 $\mu$m and in actuality measured 3.33 $\mu$m. Based off of this number we will assume that the measurement error is $\pm 3\%$.

Afterwards the sample was developed in a 3 : 1 MIBK:Isopropanol mixture for one minute. It was then placed for 20 seconds in isopropanol, which served as a stopping agent, and dried with nitrogen. In order to reuse a sample after photoresist had already been deposited it was cleaned by boiling it in acetone at 200$^\circ$C for 3 minutes. The larger more fragile membranes were left in the acetone for 5 minutes at room temperature. After the samples were taken out of the acetone bath the regular cleaning process was followed.

Once the PMMA was patterned, a 10 nm adhesion layer of chromium and 170 nm layer of 1 : 3 Au : Pd mixture was thermally evaporated onto the sample with the E-306 thermal evaporator. Gold Palladium was used as opposed to plain gold because the material holds together stronger and is therefore more suitable for the lift-off procedure. Actual thickness measurements ranged between 150 – 185 nm. The sample upon which measurements were performed had 165 nm of metal deposited on it. Liftoff was performed using NMP. The membranes were left on the hotplate at 140$^\circ$C for at least an hour, until the metal was no longer sticking where it was not meant to stick. After being boiled in NMP the samples were rinsed with isopropanol and DI water, and then dried with nitrogen.

For the final step of the membrane fabrication process, the silicone nitride membrane was etched away by electro-cyclotron resonance (ECR) etching. The fabrication process can be seen in Figure 6.1. The membrane was then glued to a PCB board (see Figure 6.5) with rubber cement, and finally the pads were wire bonded to connectors on the PCB board. This connection was a bit complicated since there are more pads than independent connectors, and the design used is shown in Figure 6.6. It should be noted that this design creates a coupling between the single beam and the three beams on side B. Side B had to be used for the measurements of the three beams since one of the three beams on side A broke.
Figure 6.1: Process Sequence.

Figure 6.2: Photograph of Jeol T300 Scanning Electron Microscope.
Figure 6.3: Beam Design.

Figure 6.4: Close Up of Beam Design.
Figure 6.5: Photograph of PCB Board.

Figure 6.6: Bonding Sketch.
6.1.2 The Final Configuration

Figures 6.7 and 6.8 show the actual sample. Figure 6.9 shows the discrepancy between the planned measurements and the actual measurements. Note that the beam-center to beam-center dimensions remain relatively close to the intended measurements, however the beams are much thicker than planned and the gaps are much smaller. Also note that in Section 2.2 we assumed symmetric gaps that are equal along the entire length of the beam and on both sides of the beam. In reality this is not necessarily true and may account for some of the discrepancy between theory and experiment.

6.2 Experimental Setup

We performed the measurements with a fiber laser. There were two set ups used for the measurements. The first, which was used only for sweep-up measurements, used a network analyzer as the external voltage source as well as a measuring instrument. For some measurements it was coupled to a DC voltage source (and a 100 MΩ resistor) as well. The second set up used a function generator as the voltage source, and a lock-in amplifier to take measurements. In both cases the laser power source and the photo-detector were attached to a coupler. The photodetector was attached to the network analyzer or the lock-in amplifier. For the single beam, the excitation was applied to the single beam of side A of the sample, and for the three-beam array, the excitation was applied to the two
Figure 6.8: Close Up of Sample.

Figure 6.9: Design versus True Measurements.
outer beams of the array on side B of the sample. A diagram of the setup can be seen in Figure 6.10. The sample was placed inside a vacuum chamber and its side view can be seen in Figure 6.11. This may be significant to the potential existence of self-excited oscillations.

We chose a network analyzer filter bandwidth of 18 Hz or 36 Hz and sweep times which varied from 60 s to 1000 s. The network analyzer measures 401 points. Therefore, the amount of time each point is measured for is equal to the sweep time divided by 401. As long as the signal remains within the filter bandwidth the voltage that exits the optical detector is averaged and this is the value that is recorded as the response to that signal. The spans we measured varied from 50 Hz to 1000 Hz. The sweep rate is equal to the span divided by the sweep time. The choice of the sweep rate is significant. Most sweep rates can be used to determine the modes, but more care must be given in determining the sweep rate when attempting to identify the damping coefficients. Too fast of a sweep rate does not properly indicate the influence of the nonlinear damping and yields inaccurate results. Too slow of a sweep rate, in addition to being inefficient, is inaccurate due to the influences of thermal drift. Our experiments for a single beam are repeatable. However, for three beams, measurements taken under the same conditions do not repeatably produce the same results. The noise floor is on the order of $2.5 \times 10^{-4}$ arbitrary units (a.u.). Arbitrary units, henceforth referred to as a.u., are a measurement of the voltage which passes through network analyzer filter and is measured by the photodetector.
6.3 Results

6.3.1 Single Beam

The results of our single beam experiments can be seen in Figures 6.12 and 6.13. Figure 6.12 plots the response magnitude versus frequency for a number of different excitation amplitudes. Figure 6.13 plots the response magnitude and phase versus frequency for an excitation magnitude of 2.94 mV. The up-down sweep obtained by the lock-in-amplifier in contrast to the sweep-up of the network analyzer gives us an idea of what the response
bandwidth is. It is apparent that qualitatively the experimental results of a single beam agree with the theory. However, we note that instead of a sharp drop depicted in Figure 4.1 the change from the trivial solution to the nontrivial solution is smooth. This can in part be explained by the asymmetry in the final configuration (see Chapter 2). The quantitative analysis is presented in section 6.4 and is based on the 2.9 mV and 3 mV experiments from Figure 6.12 and Figure 6.13.

6.3.2 Three Beams

In this subsection we present the results of the experiments done on the three beam array on side B of our sample.

a) Frequency Response

Since the experiments on the three beams were done with a laser which due to physical constraints could not achieve a very good focal distance and therefore measured an area of hundreds of microns, it can not be determined which response belongs to which beam. As such, the responses must be looked at as a whole. Figure 6.14 shows the entire frequency span in the area of the primary resonance. We see there that there are seven peaks above 0.002 a.u. Note that the frequencies of the first set of peaks vary by about 2.5%, the frequencies of the second set of peaks vary by about 1%, and the smallest frequencied peak of the 1st set is off from the largest frequencied peak of the 2nd set by about 6.2%. This graph was obtained under full vacuum (approximately $4 \times 10^{-5}$ bar). We varied the
excitation amplitude and looked at the response within this span under both full and partial (approximately $10^{-3}$ bar) vacuums. The results can be seen in Figures 6.15 and 6.18. In Figures 6.16 and 6.17 the response intensity as a function of excitation frequency is seen for excitations of 1 mV and 10 mV, respectively.

Figure 6.14: Forward Sweep Of Entire Frequency Span For Three Beams.

![Figure 6.14](image1)

Figure 6.15: Logarithmic Color Map Of Entire Frequency Span For Three Beams Subject to Different Excitation Amplitudes.

![Figure 6.15](image2)

Figure 6.19 is the continuation of Figure 6.18. It continues to measure the system response to higher excitation amplitudes. Figures 6.20 (taken at a sweep time of 199.88 s) and 6.21 show a single peak subject to a number of different excitation amplitudes.
Figure 6.16: Forward Sweep Of Entire Frequency Span For Three Beams; $V_{AC} = 1 \, mV$.

Figure 6.17: Forward Sweep Of Entire Frequency Span For Three Beams; $V_{AC} = 10 \, mV$. 
Figure 6.18: Logarithmic Color Map Of Entire Frequency Span For Three Beams in a Partial Vacuum Subject to Different Excitation Amplitudes.

Figure 6.19: Logarithmic Color Map Of Entire Frequency Span For Three Beams in a Partial Vacuum Subject to Different Excitation Amplitudes.
Figure 6.20: Forward Frequency Sweeps Of A Three Beam Response Peak For Excitation Amplitudes Ranging from 3-4 mV.

Figure 6.21: Forward Frequency Sweeps Of A Three Beam Response Peak For Excitation Amplitudes Ranging from 0.5 - 3.5 mV.
b) Sweep Up-Down

Figure 6.22 shows both forward and backward sweeps of one of the peaks seen in Figure 6.14. This peak was measured when the system was excited at a forcing amplitude of 9 mV. Figures 6.23 and 6.24 show forward and backward sweeps for another peak seen in Figure 6.14. These peaks were measured under an excitation of amplitude 4.5 mV.

\[ 9 \text{ mV} \]

\[ 4.5 \text{ mV} \]

Figure 6.22: Forward and Backward Frequency Sweeps Of A Three Beam Response Peak.

\[ \text{Intensity (a.u.)} \]

\[ \text{Phase (°)} \]

\[ \text{Frequency (Hz)} \]

\[ 10^3 \]

\[ 10^3 \]

\[ 10^3 \]

\[ 10^3 \]

Figure 6.23: Forward and Backward Frequency Sweeps Of A Three Beam Response Peak.

c) DC Bias

Figures 6.25 (under AC excitation of amplitude 3.5 mV) and 6.26 (under AC excitation of amplitude 0.4 mV) show color maps of the response of the system subject to different DC excitations. Figure 6.27 shows the current measured to be flowing through the system.
Figure 6.24: Forward and Backward Frequency Sweeps Of A Three Beam Response Peak.

versus the DC voltage applied to the system and the calculated resistance of the system (after subtracting the $100\,\text{M}\Omega$ resistance with which it is in series) versus the applied DC voltage.

Figure 6.25: Color Map Of Entire Frequency Span For Three Beams Subject to Different DC Excitations.
Figure 6.26: Color Map Of Entire Frequency Span For Three Beams Subject to Different DC Excitations.

Figure 6.27: Total Current and Device Resistance.
d) No External Excitation

In Section 1.2 we presented some background on self-excited oscillations and the photothermal effect. Recalling this and our testing setup, shown in Figure 6.11, (along with the reflective surface of the gold plated PCB board) we present in this section an interesting discovery that requires further investigation but could potentially be self-excited oscillations. Figures 6.28 and 6.29 show the possible self-excitations found to occur in the three beam system. Figures 6.30, 6.31, and 6.32 show the self-excitations measured with different laser powers when no excitation was being applied to the system.

Figure 6.28: Self-Excited Oscillation of Three Beams.
Figure 6.29: Self-Excited Oscillation of Three Beams.

Figure 6.30: Color Map Of Self-Excited Oscillations of Three Beams.
Figure 6.31: Color Map Of Self-Excited Oscillations of Three Beams.

Figure 6.32: Color Map Of Self-Excited Oscillations of Three Beams.
6.4 Comparison with Theory

6.4.1 Parameter Estimation from Experimental Data

We note that qualitatively the theoretical and experimental results for a single beam are similar. (See the theoretical results of a single beam presented in Section 4.1.1 and the experimentally obtained results of a single beam presented in Section 6.3.1.) In this section we estimate the dynamical system parameters based off of a comparison of the theoretical and experimental results. We recall that in Eq. (2.1.8) the pretension, $N$ is equal to:

$$N = \frac{EBH^3}{12\xi L^2} \quad (6.4.1)$$

We also recall (from Eq. (2.1.10)) that the time-scaling constant inverse, $\omega_s$ is equal to:

$$\omega_s = \sqrt{\frac{N}{\rho BHL^2}} \quad (6.4.2)$$

Substituting Eq. (6.4.1) into Eq. (6.4.2) yields:

$$\omega_s = \sqrt{\frac{EH^2}{12\xi \rho L^4}} \quad (6.4.3)$$

Recall (from Section 2.2) that the natural frequency of the $n$th mode in our assumed model, $\omega_n$, is equal to:

$$\omega_n = n\pi \sqrt{1 + \xi(n\pi)^2} \quad (6.4.4)$$

Thus, we deduce that the frequency, $f_n$, is equal to:

$$f_n = \frac{\omega_n \omega_s}{2\pi} = \frac{H}{4L^2} \sqrt{\frac{E(1 + \xi(n\pi)^2)}{3\xi \rho}}. \quad (6.4.5)$$

This means that the frequency of the first mode, $f_1$, is equal to:

$$f_1 = \frac{H}{4L^2} \sqrt{\frac{E(1 + \xi \pi^2)}{3\xi \rho}}. \quad (6.4.6)$$

If $\omega_j$ and $\omega_k$ are the $j$th and $k$th natural frequencies, then the value of $\tilde{\xi}$ can be estimated from Eq. (6.4.4). By dividing for $\omega_j$ by $\omega_k$ we find that:

$$\tilde{\xi} = \frac{k^2(\frac{\omega_j}{\omega_k})^2 - j^2}{\pi^2(j^4 - (\frac{\omega_j}{\omega_k})^2k^4)}. \quad (6.4.7)$$
We applied a DC voltage to the beam, and swept the drive frequency to find the resonance frequency of the first two modes. (See Appendix H.) We found that the first mode reached resonance at \( f_1 = 104,143.75 \pm 20 \text{ Hz} \) and that the second mode was excited at \( f_2 = 209,665 \pm 20 \text{ Hz} \). Taking the ratio of \( f_1 / f_2 = 2.0132 \) and substituting it back into Eq. (6.4.7) we calculate an estimated value of \( \dot{\xi} = 4.9268 \times 10^{-4} \).

We use this value of \( \dot{\xi} \) to calculate the nonlinear stiffness parameter, \( \alpha \):

\[
\alpha = \frac{\pi^2 \gamma^2 EBH}{4N(1 + \dot{\xi} \pi^2)} = \frac{3\pi^2 G^2 \dot{\xi}}{H^2(1 + \dot{\xi} \pi^2)} = 0.1038
\] (6.4.8)

In Appendix I we use Eqs. (6.4.1)-(6.4.7) to derive values for elastic modulus and density of our beams, and for the pretension. We determine there that there is a 19% disagreement between the ratio of the elastic modulus and the density obtained using the law of mixtures [24] and between the ratio obtained from the experimental data. After scaling the material property values we obtained using the law of mixtures to correct for this disagreement we determine the elastic modulus of our beams is within the range of \( 77.217 \text{ GPa} \leq E \leq 96.086 \text{ GPa} \), that the density of our beams is somewhere between \( \rho = 16922.5 \text{ kg/m}^3 \) and \( \rho = 21057.6 \text{ kg/m}^3 \), and that the initial pretension is between \( N = 4.908 \times 10^{-5} \text{ kg/s}^2 \) and \( N = 6.107 \times 10^{-5} \text{ kg/s}^2 \).

Having calculated \( \alpha \) we are left with three unknown parameters, the linear damping, \( \beta \), the nonlinear damping, \( \delta \) and the forcing magnitude, \( \nu \). We eliminate the \( \nu \) parameter by using Eq. (4.1.18) to express it in terms of \( \beta \) and the measured bandwidth, \( BW \). This gives us the relationship:

\[
\nu = \frac{1}{2} \sqrt{BW^2 + \beta^2}.
\] (6.4.9)

We measure the bandwidth from Figure 6.13. Since the experiment does not show pure parametric excitation it is unclear where the response begins and ends and where the bandwidth should be measured from. We choose two extreme possibilities, and take the average of the two and attempt to fit the experimental data to the theoretical model using these values. This gives us an idea of the range of possible bandwidths and hence the range of the relationship between the forcing parameter and the damping parameter. The range for the bandwidth which we initially chose were between \( 3.368 \times 10^{-5} \leq BW \leq 4.578 \times 10^{-5} \). (We fit the two bounds along with their average, \( 3.973 \times 10^{-5} \).) However, after comparing the theoretical results to the experimental results (see Figures J.50-J.48 in Appendix J-3), we determined that we had underestimated the bandwidth, and we reestimated it more liberally and came up with a value of \( BW = 2.189 \times 10^{-4} \).

Since the measured bandwidth was small, calculating the parameter \( \nu \) as a function of
the intended applied AC voltage, $V_{AC} = 2.94 \ mV$ yields a value of $\nu = 1.275 \times 10^{-11}$. If this were the actual value of $\nu$, then $\beta$ would have no real solution from the relationship of Eq. (6.4.9). This means that we can not base our forcing parameter value on the parallel plate model, and must further examine the forcing and linear damping parameters, as well as the nonlinear damping parameter.

In order to solve for the remaining two unknown parameters, we measure the maximum response and the saddle point for $V_{AC} = 2.9 \ mV$ and for $V_{AC} = 3 \ mV$ from Figure 6.12. We take the average of these two points so that their value can correspond to that of the bandwidth measured in Figure 6.13 for a AC voltage magnitude of $V_{AC} = 2.94 \ V$. This gives us the values $\Omega_{max} = 98884.79$, $|\varepsilon a|_{max} = 0.09755$, $\Omega*_{saddle} = 98888.92$, and $|\varepsilon a|_{saddle} = 0.09437$. Since the response magnitude is measured in arbitrary units which were obtained through an unknown nonlinear transformation, it is impossible to convert the response magnitude values to dimensional numbers. We therefore take the ratios of the magnitudes, assuming that for such a small difference in magnitude we can treat the transfer function as linear. We obtain the value \[ \frac{|\varepsilon a|_{max}}{|\varepsilon a|_{saddle}} = 1.03365. \] We now calculate this ratio analytically from Eqs. (4.1.23) and (4.1.31), substituting in Eq. (6.4.9) for $\nu$, and find that:

\[
\left( \frac{|\varepsilon a|_{max}}{|\varepsilon a|_{saddle}} \right)^2 = \frac{(2\nu - \beta)\sqrt{9\alpha^2 + \delta^2} - \beta\sqrt{9\alpha^2 + \delta^2} + 3\alpha\sqrt{BW^2 + \beta^2}}{-\beta\sqrt{9\alpha^2 + \delta^2} + 3\alpha\sqrt{BW^2 + \beta^2}} \tag{6.4.10}
\]

Note that since Eq. (6.4.10) is a squared value, and it must be positive, and as its numerator is always positive since $\sqrt{BW^2 + \beta^2} \geq \beta$, it’s denominator must be positive as well. This yields the following condition:

\[
\beta\delta < 3\alpha BW \tag{6.4.11}
\]

Taking the ratio of $\delta$ and $\beta$ as defined in Eq. (2.2.18) we find that:

\[
\frac{\delta}{\beta} = 6\left(\frac{G}{H}\right)^2 \tag{6.4.12}
\]

We recall that we assumed fixed boundary conditions but in actuality, there are stresses at the boundaries which we did not take into account and which may lead to additional boundary damping. We also assumed a constant gap, $G$, but in actuality the beams are bending and are not completely parallel to each other. A final issue worth noting is that for really small $\beta$, rather than scaling $\beta = \varepsilon^2 \hat{\beta}$ it would have to be scaled $\beta = \varepsilon^3 \hat{\beta}$ and an additional order would be necessary in the multiple scales equations. As such, we recognize that there are possible additional nonlinear damping sources (and that we may
have overestimated the influence of the linear damping) and we switch the equality in Eq. (6.4.12) to an inequality, which gives us the relationship:

\[ \frac{\delta}{\beta} \geq 6(G/H)^2 \]  

(6.4.13)

Combining Eqs. (6.4.11) and (6.4.13) and solving for \( \delta \) yields with the bounds:

\[ 6(G/H)^2 \beta \leq \delta < \frac{3\alpha BW}{\beta} \]  

(6.4.14)

In terms of the quality factor, the inequality in Eq. (6.4.14) is:

\[ \frac{6(G/H)^2}{Q} \leq \delta < 3\alpha BWQ \]  

(6.4.15)

Setting the upper and lower bounds of Eq. (6.4.14) equal to each other, and solving for \( \beta \) results in an upper bound for the linear stiffness parameter:

\[ \beta < \frac{H}{G} \sqrt{\frac{\alpha BW}{2}} \]  

(6.4.16)

Substituting the values we calculated for the nonlinear stiffness and bandwidth as well as the dimensions of our beam into Eq. (6.4.16), we come up with an upper bound for the linear damping parameter of \( \beta = 0.0012 \). This corresponds to a lower limit for the quality factor of \( Q = 830.6 \) and a nonlinear damping parameter of \( \delta = 0.0566 \).

Figures 6.33 and 6.34 plot the allowable values of the nonlinear stiffness as a function of the linear stiffness and the quality factor, respectively.

We pick a linear damping value of half the maximum allowed, \( \beta = 6.0 \times 10^{-4} \) (\( Q = 1661 \)). The corresponding nonlinear damping constant must be within the range \( 0.028 \leq \delta \leq 0.11 \), hence we pick the average of the bounds to obtain a nonlinear damping value of \( \delta = 0.07 \). Based on the relationship in Eq. (6.4.9) we calculate a forcing parameter of \( \nu = 3.2 \times 10^{-4} \) (\( V_{AC} = 14.74 \) V). To complete our parameter set we use the nonlinear stiffness parameter which we calculated in Eq. (6.4.8) of \( \alpha = 0.1038 \). Figure J.37 in Appendix J compares the experiment to the theory using these parameters. From this figure it is apparent that randomly picking a linear/nonlinear damping parameter combination does not work; an additional criteria is necessary. We retain our choice of a linear damping parameter, and pick a nonlinear damping parameter for which Eq. (6.4.10) is satisfied with the experimentally obtained data. This nonlinear damping parameter is \( \delta = 0.027 \). This value however is smaller than the lower bound and therefore can not be used. We
Figure 6.33: Upper and Lower Bounds on Nonlinear Damping as a Function of Linear Damping.

Figure 6.34: Upper and Lower Bounds on Nonlinear Damping as a Function of Quality Factor.
Figure 6.35: Frequency Response of A Single Beam System; $\delta = 0.0396$, $\alpha = 0.1038$, $\beta = 3.96 \times 10^{-4}$ ($Q = 2525$), $\nu = 2.26 \times 10^{-4}$ ($V_{AC} = 12.36V$).

attempt to maintain the nonlinear damping parameter choice of $\delta = 0.07$ and solve Eq. (6.4.10) for a linear damping parameter that satisfies the experimentally obtained data. We obtain a linear damping value of $\beta = 1.7 \times 10^{-3}$ ($Q = 5910$). This is larger than the maximum bound for $\beta$ and therefore can not be used. Hence, we conclude that randomly picking one parameter and matching the others to it does not work.

From Eq. (6.4.13) we know that for our parameters $\delta \beta \geq 47$. We choose this ratio to be $\frac{\delta}{\beta} = 100$, substitute this into Eq. (6.4.10), and obtain parameter values of $\beta = 3.96 \times 10^{-4}$ and $\delta = 0.0396$. Substituting the linear damping parameter back into Eq. (6.4.9) gives us a forcing magnitude parameter value of $\nu = 2.26 \times 10^{-4}$. Figure 6.35 shows the frequency response obtained with these parameters. It qualitatively agrees with the experiment. Appendix Appendix J shows the results of fitting this response to the data. While it’s shape and height are a good match to the data, its location is not.

Since we have two unknowns and randomly choosing one of them or their ratio is not sufficient to obtain a proper fit, we need a second equation. We initially attempted to use the ratio of the measured values of $\frac{\Omega^{*}_{sadd}}{\Omega_{max}} = 1.00004$ and tried to match it to the theory (Appendix J-1). It is evident from our analysis that while for all values of $\beta$ within the range we checked this ratio is very close to the ratio calculated theoretically,
the experiment never intersects the theory, and it remains pretty constant throughout the range of \( \delta \) values. It is therefore not possible to use it to find a best fit.

Since we could not match the ratio of the frequencies at which the saddle point and maximum response occur, we attempted to match the frequency of the maximum response \( (\Omega_{\text{max}} = 0.999954) \). Appendix J-2 shows the process of finding the parameters that fit both of these equations. The problem with this method is that the two experiments for the single beam have different bandwidths. We measure the frequency of the maximum response from the network analyzer experiment and the bandwidth from the lock-in amplifier experiment. Consequently, when we fit the network analyzer experiment to the theory, the theory plot falls in the middle of the experiment plot. We therefore decided to make our second fitting criteria the point at which the stable solution branches off from the trivial solution. The equation of this point is given in Section 4.1 in Eq. (4.1.17).

The parameter selection for the smaller bandwidths can be seen in Appendix J-3. Figure 6.36 plots the square root of Eq. (6.4.10) and Eq. (4.1.17) as a function of \( \delta \) for \( \beta = 3.096 \times 10^{-4} \). The black solid lines represent the measured experimental values. For this value of \( \beta \) we obtain the parameter set \( \delta = 4.815 \times 10^{-2}, \beta = 3.096 \times 10^{-4} (Q = 3230), \nu = 1.896 \times 10^{-4} (V_{\text{AC}} = 11.34V) \).

Notice that if we use the definition of \( \nu \) that is based on the parallel plate model to calculate the applied voltage, this number is drastically off from the voltage applied \( \approx 3\text{mV} \). Substituting the parameter values we calculated into the left side of Eq. (6.4.12) and the measured dimensions into the right side, we find that the left side is equal to 155.5. The right side is equal to 47.04. This is off by 230.6% and means that the nonlinear damping parameter is three times its lower bound. We check the influence of increasing the value of the ratio in Eq. (6.4.12) to 200 by retaining the linear damping parameter \( (\beta = 3.096 \times 10^{-4} (Q = 3230)) \) and increasing the nonlinear damping parameter to \( 6.192 \times 10^{-2} \) and by retaining the nonlinear damping parameter value \( (\delta = 0.048) \) and decreasing the linear damping parameter to \( \beta = 2.4 \times 10^{-4} (Q = 4154) \) (which results in a forcing parameter of \( \nu = 1.627 \times 10^{-4} (V_{\text{AC}} = 10.5V) \)). Increasing this ratio does not improve the fit. Keeping the new values of the damping parameters \( (\beta = 2.4 \times 10^{-4} (Q = 4154) \) and \( \delta = 0.048) \) and the old forcing parameter \( (\nu = 1.896 \times 10^{-4} (V_{\text{AC}} = 11.34V) \)) increases the bandwidth to \( 2.9 \times 10^{-4} \) and also does not improve the fit. The comparison of the theory to the experiment for these parameter sets is shown in Appendix J-4.

Figure 6.37 compares the multiple scales asymptotic approximation to the solution of the ordinary differential temporal modal equations for the set of parameters that fit the experiment. We see that the multiple scales approximation is a valid model for the behavior
Figure 6.36: Saddle and Maximum Magnitude Ratio and Branch Off Frequency for a Fixed Linear Damping Value \( (\beta = 3.096 \times 10^{-4}) \) as a Function of Nonlinear Damping \( (\delta) \). Parameters Fit for \( 2.189 \times 10^{-4} \) Bandwidth.

of the experimental system.

Figure 6.37: Multiple Scales versus Differential Equations for Experimental Parameters \( (\delta = 4.815 \times 10^{-2}, \beta = 3.096 \times 10^{-4} (Q = 3230), \nu = 1.896 \times 10^{-4} (V_{AC} = 11.34V)) \)

If we choose to ignore the data from the lock-in amplifier experiment in order to obtain a better fit for the network analyzer experiment, we assume a bandwidth of \( 2.77 \times 10^{-4} \) and obtain the parameters \( \beta = 2.14 \times 10^{-4} (Q = 4673), \delta = 0.0707, \) and \( \nu = 1.75 \times 10^{-4} (V_{AC} = 10.89 V) \).
6.4.2 Single Beam Experiment

The magnitude of the data we obtained is measured in arbitrary units whose relationship to actual physical quantities is unknown to us. We know that it is not a linear transformation. We seek a method of converting the experimental data into meaningful values. We begin by assuming that the conversion is affine (even though we know it’s not). We will define the transfer function as \( T_{data} = a \cdot |x| + b \), where \( |x| \) is the same as \( |\varepsilon a| \). We have analytical equations \( |x| \) at the maximum and saddle points from Eqs. (4.1.23) and (4.1.31). We also know the values of our experimental data at these points, namely, \( T_{max} = 0.09755 \) and \( T_{saddle} = 0.09437 \). Substituting in these values we come up with a transfer function of:

\[
T_{data} = \frac{(\delta \sqrt{9a^2 + \delta^2})^{\frac{1}{2}} (T_{max} - T_{saddle})}{2(((2\nu - \beta)\sqrt{9a^2 + \delta^2})^{\frac{1}{2}} - (6\alpha \nu - \beta \sqrt{9a^2 + \delta^2})^{\frac{1}{2}})} |x| + \\
T_{max} - 2 \frac{(\delta \sqrt{9a^2 + \delta^2})^{\frac{1}{2}} (T_{max} - T_{saddle})}{2(((2
- \beta)\sqrt{9a^2 + \delta^2})^{\frac{1}{2}} - (6\alpha \nu - \beta \sqrt{9a^2 + \delta^2})^{\frac{1}{2}})} \sqrt{\frac{2\nu - \beta}{\delta}} \tag{6.4.17}
\]

Substituting in our parameter values, we find that \( T_{data} = 1.284|x| - 1.030 \times 10^{-4} \). The original value can be backed out from this transfer function by calculating \( |x| = \frac{T_{data} - b}{a*} \), where \( b \) is the transfer function intercept and \( a* \) is the transfer function slope. This gives us the function \( |x| = 0.7787T_{data} + 8.021 \times 10^{-5} \). Figure 6.38 shows the fitting of the theory to the experiment. Note that the data from Figure 6.13 was translated so that it occurs about the same frequency about which the data of Figure 6.12 occurs.

![Figure 6.38: Comparison of Experiment to Theory with \( \delta = 4.815 \times 10^{-2}, \beta = 3.096 \times 10^{-4} \) (\( Q = 3230 \), \( \nu = 1.896 \times 10^{-4} \) (\( V_{AC} = 11.34V \)).](image-url)
Since the linear transfer function does not take into account that the response starts from zero, we considered a quadratic transfer function as well. This transformation is presented in Appendix J-5. It is not notably better than the linear transformation.

Figure 6.39 shows the fit obtained for the forwards sweep experiment when the forward/backward sweep experiment is ignored.

![Figure 6.39: Comparison of Experiment to Theory with $\beta = 2.14 \times 10^{-4}$ ($Q = 4673$), $\delta = 0.0707$, $\nu = 1.75 \times 10^{-4}$ ($V_{AC} = 10.89$ V).](image)

Figure 6.39: Comparison of Experiment to Theory with $\beta = 2.14 \times 10^{-4}$ ($Q = 4673$), $\delta = 0.0707$, $\nu = 1.75 \times 10^{-4}$ ($V_{AC} = 10.89$ V).

Note that the inverse transfer functions, calculate $x$, the scaled non-dimensional response magnitude. To obtain the temporal mode shape, $q$, we must multiply by $\frac{G}{\sqrt{2L}}$, to return to the scaled spatial beam vibration equation, $w$, we must multiply by $\frac{G}{L} \sin \pi s$, and to return to the dimensional transverse beam-string displacement, $W$, we must multiply by $\frac{G}{L^2} \sin \pi s$.

Figure 6.40 shows the dimensional response of the midpoint of the beam.

### 6.4.3 Three Beam Experiment

To compare the results of the three beam experiments with the theory, we use the parameter values we calculated for $\beta = 3.096 \times 10^{-4}$, $\alpha = 0.1038$, and $\delta = 4.815 \times 10^{-2}$ in Section 6.4.1, estimate the bandwidth from Figure 6.23 to be $8.090 \times 10^{-5}$, and calculate a forcing amplitude parameter value of $\nu = 1.600 \times 10^{-4}$ $V_{AC} = 10.42$ V. The individual responses of each of the three beams are shown in Figure 6.41. We then take the root mean square of the responses of the three beams and compare it to the experiment. The results
Figure 6.40: Comparison of Experiment to Theory with $\beta = 2.14 \times 10^{-4}$ ($Q = 4673$), $\delta = 0.0707$, $\nu = 1.75 \times 10^{-4}$ ($V_{AC} = 10.89$ V).

are shown in Figure 6.42. The three beam experiments do not agree with the theory. This can be for a number of reasons which will be discussed in the next section.
Figure 6.41: Comparison of Experiment to Theory for 3 Beams $\delta = 4.815 \times 10^{-2}$, $\beta = 3.096 \times 10^{-4}$ ($Q = 3230$), $\nu = 1.6 \times 10^{-4}$ ($V_{AC} = 10.42V$).

Figure 6.42: Comparison of Experiment to Theory for 3 Beams $\delta = 4.815 \times 10^{-2}$, $\beta = 3.096 \times 10^{-4}$ ($Q = 3230$), $\nu = 1.6 \times 10^{-4}$ ($V_{AC} = 10.42V$).
6.5 Discussion

The single beam experiments qualitatively match the theory. They match it quantitatively as well, but the experiments themselves do not agree with each other (in terms of bandwidths, amplitudes, and resonance frequencies) and make it difficult to fit the theory to the experiment. Nonetheless, we have managed with a bit of playing around to make the theory quantitatively fit the experiment. In the process of fitting the theoretical model to the experimental data we discovered that if a larger nonlinear stiffness parameter is required it will be primarily governed by the frequency ratio (e.g. $\tilde{\xi}$) and the ratio of $\frac{G}{\mu}$.

There are a number of issues that help explain why the three-beam theory does not agree with the experiments. First of all, we assumed degeneracy when we modeled the system. However, in order for this assumption to be accurate, the bandwidth of a response must be greater than the difference between frequencies of it and the response of the next beam. This does not hold true for our case, as can be seen in Figure 6.15. In a vacuum of $4\times10^{-5}$ bar the Q-factor is very large and therefore the response bandwidth is very narrow. Decreasing the Q-factor (by reducing the vacuum) (hence increasing the bandwidth) can help improve the accuracy of the degeneracy assumption. Second, we recall that the microbeam width was much smaller than its height ($B = 165 \text{ nm}$, $H = 1.75 \mu\text{m}$). Thus, the system may exhibit three dimensional effects, including modal energy transfer, as the directly excited first and second transverse modes (e.g. primary and principle parametric resonance) correspond to the tenth and higher lateral bending beam-string modes.

In Figure 6.19 we see some agreement with the expected theoretical results, such as the response threshold and the wider bandwidth. Also apparent in Figure 6.15 is that although the measurements were taken of an excitation on a three beam array, there are more than three responses. Even taking into account the fact that the single beam on side B of the device (where these measurements are taken) is coupled to the top beam of the three beams would only explain four peaks, but there are more than four peaks. Since capacitance decays slowly as a function of distance, the electrically floating beams from side A of the device may be vibrating as well, and this can account for the additional peaks.

Another issue which was not considered in the theory but is evident from the experimental analysis (see Figures 6.30-6.32), is the influence of temperature on resonance frequency. The final issue which we choose to discuss is the influence of applying DC voltage to the system. Although the theory did not take into account external excitation, we wanted to attempt to increase the microbeam interaction and coexistence of solutions through the application of a DC voltage. However, as can be seen in Figures 6.25 and 6.26, contrary
to logic, the bias had absolutely no effect on the resonance. This is because the sample apparently suffered from current leakage as can be seen in Figure 6.27. As such, no experiments could be done on the sample using a DC bias. This also prevented us from further investigating the existence of self-excited oscillations.
Chapter 7

Closing Remarks

7.1 Summary

In this research we have succeeded in deriving a continuum mechanics based model for a parametrically excited MEMS array that includes both nonlinear stiffness and nonlinear damping. We have constructed a reduced order dynamical system, and conducted a stability analysis that included finding the system’s equilibrium points and natural frequencies (for both a system consisting of a single beam and of a 3-beam array). We analyzed the system asymptotically, using the Multiple Scales method, and numerically, using MATLAB’s ODE45 solver. We came up with an analytical solution for the response amplitude and phase for a single beam, and calculated values for the maximum response amplitude and frequency, the saddle point amplitude and frequency, and the bandwidth. We compared these values for a number of different parameter sets to the results obtained from using ODE45 on the modal dynamical system. We also derived an analytical solution for the bandwidth of the response of a beam from the three beam array. We used the ordered equations for three beams in the MATCONT toolbox of MATLAB to numerically find the Multiple Scales solution for a three beam array, and compared this with the numerical solution of the differential equations obtained by the ODE45 solver. We have discovered a complex bifurcation structure with coexisting periodic solutions. We manufactured an array that consisted of single and three element systems, electrostatically excited it by applying an AC voltage, and measured its response using both a spectrum analyzer and a lock-in amplifier that allowed us to perform frequency sweeps in both directions. Experimental analysis of the array did not reveal coexisting solutions, possibly due to a large gap and inability to apply DC voltage due to leakage. We discovered possible self-excited oscillations due to induced laser irradiation. The single beam experimental results quali-
tatively agreed with the theoretical results, and we fitted its data to the Multiple scales equations to come up with a set of parameters. The three beam results however, did not agree with the theoretical results.

7.2 Conclusions

We conclude that we have derived an accurate nonlinear continuum based model for the behavior of a single beam that accounts for experimentally determined behavior, and can be quantitatively fit to the experimental results. We believe that our model is also accurate for a three beam array, however, due to several complications, our experimental results for three beams are inconclusive. They are not what we expected them to be and we do not know how to explain them. We revealed what could potentially be self-excited oscillations, but at present time this can not be confirmed. We discovered that the stability of the system is dependent solely on the forcing amplitude, and we came up with the critical values of this parameter. We also discovered that for small forcing amplitude, 1 : 1 : 1 internal resonance can be assumed. We have found that the multiple scales approximation is a good approximation for these equations. We numerically found coexisting in-phase/out-of-phase solutions as well as a quasiperiodic solution for the three beam system subject to a linearized electrostatic force. We determined that the truncated linearized parametric excitation is sufficient to demonstrate loss of orbital stability and periodicity. We found upper and lower bounds for the nonlinear damping parameter and an upper bound for the linear damping parameter. We discovered that the nonlinear stiffness parameter is a function of the geometry of the array and the ratio of the natural frequencies of the first two modes of the beam. Finally, we conclude that the three beam experiments should be repeated under better conditions, which will be discussed in the next section.

7.3 Recommendations for Future Research

It is recommended that further research be done in the following areas:

- Equations should be derived of which take into account DC voltage. (This may require the use of multiple modes in the Galerkin reduction.)
- Equations should be derived which account for the presence of self-excited oscillations.
• Quasiperiodic solutions should be looked for in the forcing frequency range of
\[
1.0015\pi \sqrt{\frac{N(1+\xi \pi^2)}{\rho AL^2}} \leq \omega_{AC} \leq 1.0035\pi \sqrt{\frac{N(1+\xi \pi^2)}{\rho AL^2}}
\]
for a forcing amplitude of \( V_{AC} = 161505 \sqrt{\frac{NG^3(1+\xi \pi^2)}{B}} \) V for a system with a quality factor of \( Q = 500 \).

• Further experiments should be done using a DC excitation so that microbeam interaction can be deduced and so that self-excitations can be validated.

• Measurements should be taken in a setup that allows for the identification of individual element behavior (either with much better focal distance or with an e-beam).

• Finally, we recommend that measurements be taken in a controlled partial vacuum (to decrease the quality factor) and that the value of the nonlinear stiffness parameter be controlled by the choice of dimensions of the gap width and the height of the beam.
Bibliography


Appendix A  Equilibrium Stability Analysis of Single Beam

The Eigenvalues of the unforced (ν = 0) single beam are:

\[ \lambda_{1,2} = -\frac{\beta}{2} \pm \sqrt{\frac{\beta^2}{4} - 1} = -\frac{\beta}{2} \pm i\sqrt{1 - \frac{\beta^2}{4}} \]  \hspace{1cm} (A-1)

By analyzing these eigenvalues, it is found that for \( \beta \geq 2 \) the point is a stable sink and for \( \beta < 2 \) this point is a stable spiral.

The Eigenvalues for the trivial equilibrium point of a single beam with with forcing bias (ν ≠ 0) are:

\[ \lambda_{1,2} = -\frac{\beta}{2} \pm \sqrt{\frac{\beta^2}{4} - 1 + 4\nu} \]  \hspace{1cm} (A-2)

This means that if \( \nu < \frac{1}{4} \) and \( \beta < 2\sqrt{1 - 4\nu} \) the point is a stable spiral. Otherwise, if \( \nu \leq \frac{1}{4} \) but \( \beta > 2\sqrt{1 - 4\nu} \) the point is a stable sink node. Finally, if \( \nu > \frac{1}{4} \), the point is a saddle point.
Appendix B  Asymptotic Analysis of Single Beam

Substituting $\varepsilon^2 \hat{\beta} = \beta$ and Eq. (4.1.1) into Eq. (2.2.22) results in:

\[
\sum_{i=1}^{3} \varepsilon^i D_0^2 x_i + 2 \sum_{i=1}^{2} \varepsilon^{i+1} D_0 D_1 x_i + \varepsilon^3 (D_1^2 + 2D_0 D_2) x_1 + \varepsilon^3 \hat{\beta} D_0 x_1 + \\
\sum_{i=1}^{3} \varepsilon^i x_i + \delta \varepsilon x_1^2 \varepsilon D_0 x_1 + \varepsilon^3 \alpha x_1^3 + O(\varepsilon^4) = \varepsilon^3 \hat{\nu} [1 + \cos (2\Omega T_0)] x_1 + O(\varepsilon^4) (B-3)
\]

Substituting $x_2 = 0$ and Eq. (4.1.5), as well as the detuning relationship, $\varepsilon^2 \sigma = \Omega - 1$, into Eq. (4.1.4) results in the following equation:

\[
D_0^2 x_3 + x_3 = 2i[-D_2 A(T_2) \exp (iT_0) + D_2 \bar{A}(T_2) \exp (-iT_0)] + \\
\hat{\beta} i[-A(T_2) \exp (iT_0) + \bar{A}(T_2) \exp (-iT_0)] + \\
\delta i[-A^3(T_2) \exp (3iT_0) - A^2(T_2) \bar{A}(T_2) \exp (iT_0)] + \\
A(T_2) \bar{A}^2(T_2) \exp (-iT_0) + \bar{A}^3(T_2) \exp (-3iT_0)] - \\
\alpha[A^3(T_2) \exp (3iT_0) + 3A^2(T_2) \bar{A}(T_2) \exp (iT_0) + \\
3A(T_2) \bar{A}^2(T_2) \exp (-iT_0) + \bar{A}^3(T_2) \exp (-3iT_0)] + \\
2\hat{\nu}[2A(T_2) \exp (iT_0) + A(T_2) \exp (i(2\sigma T_2 + 3T_0))] + \\
A(T_2) \exp (-2i\sigma T_2) \exp (-iT_0) + 2\bar{A}(T_2) \exp (-iT_0) + \\
\bar{A}(T_2) \exp (2i\sigma T_2) \exp (iT_0) + A(T_2) \exp (-i(2\sigma T_2 + 3T_0)) (B-4)
\]

Setting the $\exp (-iT_0)$ terms from Eq. (B-4) equal to zero results in the following equation:

\[
2i D_2 \bar{A} + \hat{\beta} \bar{A} + \delta i AA^2 - 3\alpha AA^2 + 2\hat{\nu}[A \exp (-2i\sigma T_2) + 2\bar{A}] = 0 (B-5)
\]

Appendix C  Single Beam Numerical Analysis

In Figure C.1, the dots represent the Cartesian solution obtained by MATCONT, the asterisks represent the polar solution obtained by MATCONT, the o-s represent the Cartesian solution obtained by MATLAB using the ode45 command, and the x-es represent the solution to the original ordinary differential Eq. (2.2.22) obtained by MATLAB use the ode45 command. The parameters are the same as those of Figure 4.1.
Figure C.1: Frequency Response of A Single Beam System; $\delta = 2$, $\alpha = 4$, $\beta = 0.002$, $\nu = 0.002$ ($V_{AC} = 36.83V$).

Appendix D  Asymptotic Analysis of Three Beams

Substituting $\beta = \varepsilon^2 \hat{\beta}$ and $\nu = \varepsilon^2 \hat{\nu}$ into the 3 beam equations (2.2.23)–(2.2.25) results in:

$$\sum_{i=1}^{3} \left[ \varepsilon^i(D_0^2 x_{1i} + x_{1i}) + 2\varepsilon^{i+1}D_0D_1 x_{1i} + \varepsilon^{i+2}(D_1^2 x_{1i} + 2D_0D_2 x_{1i} + \hat{\beta}D_0 x_{1i}) + O(\varepsilon^{i+3}) \right] +$$

$$\delta \left[ \sum_{i=1}^{3} \varepsilon^i x_{1i} \right]^2 \sum_{i=1}^{3} \left[ \varepsilon^i D_0 x_{1i} + O(\varepsilon^{i+1}) \right] + \alpha \left[ \sum_{i=1}^{3} \varepsilon^i x_{1i} \right]^3 =$$

$$\varepsilon^2 \hat{\nu}(1 + \cos(2\Omega T_0))\{2 \sum_{i=1}^{3} \varepsilon^i x_{1i} - \sum_{i=1}^{3} \varepsilon^i x_{2i} \} +$$

$$3(-2 \sum_{i=1}^{3} \varepsilon^i x_{2i} \sum_{i=1}^{3} \varepsilon^i x_{1i} + \left[ \sum_{i=1}^{3} \varepsilon^i x_{2i} \right]^2) + 4(2 \sum_{i=1}^{3} \varepsilon^i x_{1i} \right]^3$$

$$-3 \sum_{i=1}^{3} \varepsilon^i x_{2i} \sum_{i=1}^{3} \varepsilon^i x_{1i} \right]^2 + 3 \left[ \sum_{i=1}^{3} \varepsilon^i x_{2i} \right]^2 \sum_{i=1}^{3} \varepsilon^i x_{1i} - \sum_{i=1}^{3} \varepsilon^i x_{2i} \right]^3 \right}\} \quad \text{(D-6)}$$

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Equations (D-6)–(D-8) are simplified to:

\[ \sum_{i=1}^{3} [\varepsilon^i(D_0^2 x_{2i} + x_{2i}) + 2\varepsilon^{i+1} D_0 D_1 x_{2i} + \\
\varepsilon^{i+2}(D_1^2 x_{2i} + 2D_0 D_2 x_{2i} + \dot{\beta} D_0 x_{2i}) + O(\varepsilon^{i+3})] + \\
\delta [\sum_{i=1}^{3} \varepsilon^i x_{2i}]^2 \sum_{i=1}^{3} [\varepsilon^i D_0 x_{2i} + O(\varepsilon^{i+1})] + \alpha [\sum_{i=1}^{3} \varepsilon^i x_{2i}]^3 = \\
\varepsilon^2 \hat{\nu}(1 + \cos (2\Omega T_0)) [2(2 \sum_{i=1}^{3} \varepsilon^i x_{2i} - \sum_{i=1}^{3} \varepsilon^i x_{1i} - \sum_{i=1}^{3} \varepsilon^i x_{3i}) + \\
3(2 \sum_{i=1}^{3} \varepsilon^i x_{2i} + 3 \sum_{i=1}^{3} \varepsilon^i x_{1i} - 2 \sum_{i=1}^{3} \varepsilon^i x_{3i} - [\sum_{i=1}^{3} \varepsilon^i x_{1i}]^2 + [\sum_{i=1}^{3} \varepsilon^i x_{3i}]^2]) + \\
4(2 \sum_{i=1}^{3} \varepsilon^i x_{2i})^3 - 3[\sum_{i=1}^{3} \varepsilon^i x_{2i}]^2 \sum_{i=1}^{3} \varepsilon^i x_{1i} - 3 \sum_{i=1}^{3} \varepsilon^i x_{3i} \sum_{i=1}^{3} \varepsilon^i x_{2i})^2 + \\
3[\sum_{i=1}^{3} \varepsilon^i x_{2i}]^3 - 3[\sum_{i=1}^{3} \varepsilon^i x_{1i}]^2 + 3[\sum_{i=1}^{3} \varepsilon^i x_{3i}]^2 \sum_{i=1}^{3} \varepsilon^i x_{2i} - [\sum_{i=1}^{3} \varepsilon^i x_{1i}]^3 - [\sum_{i=1}^{3} \varepsilon^i x_{3i}]^3] \] (D-7)

\[ \sum_{i=1}^{3} [\varepsilon^i(D_0^2 x_{3i} + x_{3i}) + 2\varepsilon^{i+1} D_0 D_1 x_{3i} + \\
\varepsilon^{i+2}(D_1^2 x_{3i} + 2D_0 D_2 x_{3i} + \dot{\beta} D_0 x_{3i}) + O(\varepsilon^{i+3})] + \\
\delta [\sum_{i=1}^{3} \varepsilon^i x_{3i}]^2 \sum_{i=1}^{3} [\varepsilon^i D_0 x_{3i} + O(\varepsilon^{i+1})] + \alpha [\sum_{i=1}^{3} \varepsilon^i x_{3i}]^3 = \\
\varepsilon^2 \hat{\nu}(1 + \cos (2\Omega T_0)) [2(2 \sum_{i=1}^{3} \varepsilon^i x_{3i} - \sum_{i=1}^{3} \varepsilon^i x_{2i}) + \\
3(2 \sum_{i=1}^{3} \varepsilon^i x_{2i} + 3 \sum_{i=1}^{3} \varepsilon^i x_{3i} - [\sum_{i=1}^{3} \varepsilon^i x_{2i}]^2 + [\sum_{i=1}^{3} \varepsilon^i x_{3i}]^2] + \\
4(2 \sum_{i=1}^{3} \varepsilon^i x_{3i})^3 - 3[\sum_{i=1}^{3} \varepsilon^i x_{3i}]^2 \sum_{i=1}^{3} \varepsilon^i x_{2i} - 3 \sum_{i=1}^{3} \varepsilon^i x_{3i} \sum_{i=1}^{3} \varepsilon^i x_{2i})^2 + \\
3[\sum_{i=1}^{3} \varepsilon^i x_{2i}]^3 - 3[\sum_{i=1}^{3} \varepsilon^i x_{3i}]^2 + 3[\sum_{i=1}^{3} \varepsilon^i x_{2i}]^2 \sum_{i=1}^{3} \varepsilon^i x_{3i} - [\sum_{i=1}^{3} \varepsilon^i x_{2i}]^3] \] (D-8)

Equations (D-6)–(D-8) are simplified to:

\[ \varepsilon[D_0^2 x_{11} + x_{11}] + \varepsilon^2[D_0^2 x_{12} + x_{12} + 2D_0 D_1 x_{11}] + \\
\varepsilon^3[D_0^2 x_{13} + x_{13} + 2D_0 D_1 x_{12} + D_1^2 x_{11} + 2D_0 D_2 x_{11} + \\
\dot{\beta} D_0 x_{11} + \delta x_1^2 D_0 x_{11} + \alpha x_3^2 x_{11}] + O(\varepsilon^4) = \\
\varepsilon^3 [\hat{\nu}[2 + \exp (2\Omega T_0) + \exp (-2\Omega T_0)][2x_{11} + x_{21}]] + O(\varepsilon^4) \] (D-9)

\[ \varepsilon[D_0^2 x_{21} + x_{21}] + \varepsilon^2[D_0^2 x_{22} + x_{22} + 2D_0 D_1 x_{21}] + \\
\varepsilon^3[D_0^2 x_{23} + x_{23} + 2D_0 D_1 x_{22} + D_1^2 x_{21} + 2D_0 D_2 x_{21} + \\
\varepsilon[D_0^2 x_{31} + x_{31}] + \varepsilon^2[D_0^2 x_{32} + x_{32} + 2D_0 D_1 x_{31}] + \\
\varepsilon^3[D_0^2 x_{33} + x_{33} + 2D_0 D_1 x_{32} + D_1^2 x_{31} + 2D_0 D_2 x_{31} +}
\[
\dot{\beta}D_0 x_{21} + \delta x_{21}^2 D_0 x_{21} + \alpha x_{21}^3 + O(\epsilon^4) = \\
\epsilon^3 \nu[2 + \exp(2\Omega T_0) + \exp(-2\Omega T_0)](2x_{21} - x_{11} - x_{31}) + O(\epsilon^4) \tag{D-10}
\]

\[
\epsilon[D_0^2 x_{31} + x_{31}] + \epsilon^2[D_0^2 x_{32} + x_{32} + 2D_0 D_1 x_{31}] + \\
\epsilon^3[D_0^2 x_{33} + x_{33} + 2D_0 D_1 x_{32} + D_3^2 x_{31} + 2D_0 D_2 x_{31} + \\
\dot{\beta}D_0 x_{31} + \delta x_{31}^2 D_0 x_{31} + \alpha x_{31}^3 + O(\epsilon^4) = \\
\epsilon^3 \nu[2 + \exp(2\Omega T_0) + \exp(-2\Omega T_0)](2x_{31} - x_{21}) + O(\epsilon^4) \tag{D-11}
\]

Substituting the detuning relation \(\epsilon^2 \sigma = \Omega - 1\), Eq. (4.1.5), and \(x_{n2} = 0\) into Eqs. (4.2.43) results in:

\[
D_0^2 x_{13} + x_{13} = -D_1^2[A_1(T_2) \exp(iT_0) + \bar{A}_1(T_2) \exp(-iT_0)] - \\
2D_0 D_2[A_1(T_2) \exp(iT_0) + \bar{A}_1(T_2) \exp(-iT_0)] \\
-\beta D_0[A_1(T_2) \exp(iT_0) + \bar{A}_1(T_2) \exp(-iT_0)] \\
-\delta[A_1(T_2) \exp(iT_0) + \bar{A}_1(T_2) \exp(-iT_0)]^2 D_0[A_1(T_2) \exp(iT_0) + \\
\bar{A}_1(T_2) \exp(-iT_0)] - \alpha[A_1(T_2) \exp(iT_0) + \bar{A}_1(T_2) \exp(-iT_0)]^3 + \\
\nu[2 + \exp(2i(\sigma T_2 + T_0))] + \\
\exp(-2i(\sigma T_2 + T_0)])[2[A_1(T_2) \exp(iT_0) + \\
\bar{A}_1(T_2) \exp(-iT_0)] - [A_2(T_2) \exp(iT_0) + \bar{A}_2(T_2) \exp(-iT_0)] = \\
2i[-D_2 A_1 \exp(iT_0) + D_2 \bar{A}_1 \exp(-iT_0)] + \\
\beta i[-A_1 \exp(iT_0) + \bar{A}_1 \exp(-iT_0)] + \\
\delta i[-A_1^3 \exp(3iT_0) - A_1^2 \bar{A}_1 \exp(iT_0) + A_1 \bar{A}_1^2 \exp(-iT_0) + \\
\bar{A}_1^3 \exp(-3iT_0)] - \alpha[A_1^3 \exp(3iT_0) + 3A_1^2 \bar{A}_1 \exp(iT_0) + \\
3A_1 \bar{A}_1^2 \exp(-iT_0) + \bar{A}_1^3 \exp(-3iT_0)] + \\
\nu[2(2A_1 - A_2) \exp(iT_0) + [2A_1 - A_2] \exp(i(2\sigma T_2 + 3T_0))] + \\
[2A_1 - A_2] \exp(-2i\sigma T_2) \exp(-iT_0) + 2[2A_1 - A_2] \exp(-iT_0) + \\
[2A_1 - A_2] \exp(2i\sigma T_2) \exp(iT_0) + [2A_1 - A_2] \exp(-i(2\sigma T_2 + 3T_0))] \tag{D-12}
\]

\[
D_0^2 x_{23} + x_{23} = -D_1^2[A_2(T_2) \exp(iT_0) + \bar{A}_2(T_2) \exp(-iT_0)] - \\
2D_0 D_2[A_2(T_2) \exp(iT_0) + \bar{A}_2(T_2) \exp(-iT_0)] \\
-\beta D_0[A_2(T_2) \exp(iT_0) + \bar{A}_2(T_2) \exp(-iT_0)] \\
-\delta[A_2(T_2) \exp(iT_0) + \bar{A}_2(T_2) \exp(-iT_0)]^2 D_0[A_2(T_2) \exp(iT_0) + \\
\bar{A}_2(T_2) \exp(-iT_0)] \exp(-2\Omega T_0)] + \\
\exp (-2\Omega T_0) \exp(iT_0) + \exp(-2\Omega T_0)](2x_{23} - x_{31} - x_{21}) + O(\epsilon^4) \tag{D-13}
\]
\[ \tilde{A}_2(T_2) \exp (-iT_0) - \alpha [A_2(T_2) \exp (iT_0) + \tilde{A}_2(T_2) \exp (-iT_0)]^3 +
2\tilde{\nu}(2 + \exp (2i(\sigma T_2 + T_0)) + \exp (-2i(\sigma T_2 + T_0)))(A_2(T_2) \exp (iT_0) +
\tilde{A}_2(T_2) \exp (-iT_0)] - [A_1(T_2) \exp (iT_0) + \tilde{A}_1(T_2) \exp (-iT_0)] -
[A_3(T_2) \exp (iT_0) + \tilde{A}_3(T_2) \exp (-iT_0)] =
2i[-D_2A_2 \exp (iT_0) + D_2\tilde{A}_2 \exp (-iT_0)] +
\hat{\beta}i[-A_2 \exp (iT_0) + \tilde{A}_2 \exp (-iT_0)] + \delta i[-A_3^2 \exp (3iT_0) +
-A_2^2A_2 \exp (iT_0) + A_2A_2^2 \exp (-iT_0) + \tilde{A}_3^2 \exp (-3iT_0)] +
\nu(2[2A_2 - A_1 - A_3] \exp (iT_0) +
[2A_2 - A_1 - A_3] \exp (-i(2\sigma T_2 + 3T_0))] (D-13)

\[ D^2_0 x_{33} + x_{33} = -D^2_1 [A_3(T_2) \exp (iT_0) + \tilde{A}_3(T_2) \exp (-iT_0)] -
2D_0D_2 [A_3(T_2) \exp (iT_0) + \tilde{A}_3(T_2) \exp (-iT_0)] -
\hat{\beta}D_0 [A_3(T_2) \exp (iT_0) + \tilde{A}_3(T_2) \exp (-iT_0)] -
\delta [A_3(T_2) \exp (iT_0) + \tilde{A}_3(T_2) \exp (-iT_0)]^2 D_0 [A_3(T_2) \exp (iT_0) +
\tilde{A}_3(T_2) \exp (-iT_0)] - \alpha [A_3(T_2) \exp (iT_0) + \tilde{A}_3(T_2) \exp (-iT_0)]^3 +
2\tilde{\nu}(2 + \exp (2i(\sigma T_2 + T_0)))(A_3(T_2) \exp (iT_0) +
A_3(T_2) \exp (-iT_0)] - [A_2(T_2) \exp (iT_0) + \tilde{A}_2(T_2) \exp (-iT_0)] =
2i[-D_2A_3 \exp (iT_0) + D_2\tilde{A}_3 \exp (-iT_0)] + \hat{\beta}i[-A_3 \exp (iT_0) +
\tilde{A}_3 \exp (-iT_0)] + \delta i[-A_3^3 \exp (3iT_0) - A_3^2 \tilde{A}_3 \exp (iT_0) +
A_3A_3^2 \exp (-iT_0) + \tilde{A}_3^3 \exp (-3iT_0)] - \alpha [A_3^3 \exp (3iT_0) +
3A_3^2A_3 \exp (iT_0) + 3A_3A_3^2 \exp (-iT_0) + \tilde{A}_3^3 \exp (-3iT_0)] +
\nu(2[2A_3 - A_2] \exp (iT_0) + [2A_3 - A_2] \exp (i(2\sigma T_2 + 3T_0)) +
[2A_3 - A_2] \exp (-2i\sigma T_2) \exp (-iT_0)] + [2A_3 - A_2] \exp (-i(2\sigma T_2 + 3T_0))] +
[2\tilde{A}_3 - \tilde{A}_2] \exp (2i\sigma T_2) \exp (iT_0) + [2\tilde{A}_3 - \tilde{A}_2] \exp (-i(2\sigma T_2 + 3T_0))) (D-14) \]
The complex conjugate equations for a 3-beam array look as follows:

\[
\begin{align*}
2iD_2 \bar{A}_1 + \beta i \bar{A}_1 + \delta i A_1 A_1^2 & \\
-3\alpha A_1 \bar{A}_1^2 + \nu (\{2A_1 - A_2\} \exp (-2i\sigma T_2) + 2\{2\bar{A}_1 - \bar{A}_2\}) & = 0 \\
2iD_2 \bar{A}_2 + \beta i \bar{A}_2 + \delta i A_2 \bar{A}_2^2 & \\
-3\alpha A_2 \bar{A}_2^2 + \nu (\{2A_2 - A_1 - A_3\} \exp (-2i\sigma T_2) + 2\{2\bar{A}_2 - \bar{A}_1 - \bar{A}_3\}) & = 0 \\
2iD_2 \bar{A}_3 + \beta i \bar{A}_3 + \delta i A_3 \bar{A}_3^2 & \\
-3\alpha A_3 \bar{A}_3^2 + \nu (\{2A_3 - A_2\} \exp (-2i\sigma T_2) + 2\{2\bar{A}_3 - \bar{A}_2\}) & = 0
\end{align*}
\]  

(D-15)

Appendix E  Additional MATCONT Solutions to Cartesian Evolution Equations

Figure E.2: Frequency Response of First and Third Beams of a 3-Beam Array; \(\delta = 0, \alpha = 4, \beta = 0.002 \ (Q = 500), \nu = 0.002 \ (V_{AC} = 36.83V)\).
Figure E.3: Frequency Response of Middle Beam of a 3-Beam Array; $\delta = 0$, $\alpha = 4$, $\beta = 0.002$ ($Q = 500$), $\nu = 0.002$ ($V_{AC} = 36.83V$).

Figure E.4: Frequency Response of First and Third Beams of a 3-Beam Array; $\delta = 0.14$, $\alpha = 0.186$, $\beta = 0.002$ ($Q = 500$), $\nu = 0.002$ ($V_{AC} = 36.83V$).
Figure E.5: Frequency Response of Middle Beam of a 3-Beam Array; $\delta = 0.14$, $\alpha = 0.186$, $\beta = 0.002$ ($Q = 500$), $\nu = 0.002$ ($V_{AC} = 36.83V$).

Figure E.6: Frequency Response of First and Third Beams of a 3-Beam Array; $\delta = 0.14$, $\alpha = 0.186$, $\beta = 0.002$ ($Q = 500$), $\nu = 0.001125$ ($V_{AC} = 27.62V$).
Figure E.7: Frequency Response of Middle Beam of a 3-Beam Array; $\delta = 0.14$, $\alpha = 0.186$, $\beta = 0.002$ ($Q = 500$), $\nu = 0.001125$ ($V_{AC} = 27.62V$).

Figure E.8: Frequency Response of First and Third Beams of a 3-Beam Array; $\delta = 0.14$, $\alpha = 0.186$, $\beta = 0.002$ ($Q = 500$), $\nu = 0.001$ ($V_{AC} = 26.04V$).
Figure E.9: Frequency Response of Middle Beam of a 3-Beam Array; $\delta = 0.14$, $\alpha = 0.186$, $\beta = 0.002$ ($Q = 500$), $\nu = 0.001$ ($V_{AC} = 26.04V$).

Figure E.10: Frequency Response of First and Third Beams of a 3-Beam Array; $\delta = 0.14$, $\alpha = 0.186$, $\beta = 0.002$ ($Q = 500$), $\nu = 0.0009$ ($V_{AC} = 24.71V$).
Figure E.11: Frequency Response of Middle Beam of a 3-Beam Array; $\delta = 0.14$, $\alpha = 0.186$, $\beta = 0.002$ ($Q = 500$), $\nu = 0.0009$ ($V_{AC} = 24.71 V$).

Figure E.12: Frequency Response of First and Third Beams of a 3-Beam Array; $\delta = 0.14$, $\alpha = 0.186$, $\beta = 0.002$ ($Q = 500$), $\nu = 0.0008$ ($V_{AC} = 23.29 V$).
Figure E.13: Frequency Response of Middle Beam of a 3-Beam Array; $\delta = 0.14$, $\alpha = 0.186$, $\beta = 0.002$ ($Q = 500$), $\nu = 0.0008$ ($V_{AC} = 23.29V$).

Figure E.14: Frequency Response of First and Third Beams of a 3-Beam Array; $\delta = 0.14$, $\alpha = 0.186$, $\beta = 0.002$ ($Q = 500$), $\nu = 0.0007$ ($V_{AC} = 21.79V$).
Figure E.15: Frequency Response of Middle Beam of a 3-Beam Array; $\delta = 0.14$, $\alpha = 0.186$, $\beta = 0.002$ ($Q = 500$), $\nu = 0.0007$ ($V_{AC} = 21.79V$).
Figure E.16: Frequency Response of First and Third Beams of a 3-Beam Array; $\delta = 2$, $\alpha = 4$, $\beta = 0.002$ ($Q = 500$), $\nu = 0.0065714$ ($V_{AC} = 66.76V$).
Figure E.17: Frequency Response of Middle Beam of a 3-Beam Array; $\delta = 2$, $\alpha = 4$, $\beta = 0.002$ ($Q = 500$), $\nu = 0.0065714$ ($V_{AC} = 66.76V$).

Figure E.18: Frequency Response of First and Third Beams of a 3-Beam Array; $\delta = 2$, $\alpha = 4$, $\beta = 0.0021428$ ($Q = 466.7$) ($Q = 466.7$), $\nu = 0.0080714$ ($V_{AC} = 73.98V$).
Figure E.19: Frequency Response of Middle Beam of a 3-Beam Array; $\delta = 2$, $\alpha = 4$, $\beta = 0.0021428$ ($Q = 466.7$), $\nu = 0.0080714$ ($V_{AC} = 73.98V$).

Figure E.20: Frequency Response of First and Third Beams of a 3-Beam Array; $\delta = 2$, $\alpha = 4$, $\beta = 0.0045714$ ($Q = 218.8$), $\nu = 0.0077857$ ($V_{AC} = 72.66V$).
Figure E.21: Frequency Response of Middle Beam of a 3-Beam Array; $\delta = 2$, $\alpha = 4$, $\beta = 0.0045714$ ($Q = 218.8$), $\nu = 0.0077857$ ($V_{AC} = 72.66V$).
Appendix F  MATCONT Cartesian Components

The following figures show the Cartesian components obtained by MATCONT of the solution shown in Figures 4.8 and 4.9:

Figure F.22: P-Component of Frequency Response For First and Third Beams of a 3-Beam Array; $\delta = 2$, $\alpha = 4$, $\beta = 0.002$, $\nu = 0.002$ ($V_{AC} = 36.83V$).

The following figures show the Cartesian components obtained by MATCONT of the solution shown in Figures E.16 and E.17:
Figure F.23: Q-Component of Frequency Response of First and Third Beams of a 3-Beam Array; $\delta = 2$, $\alpha = 4$, $\beta = 0.002$, $\nu = 0.002$ ($V_{AC} = 36.83V$).
Figure F.24: P-Component of Frequency Response For Middle Beam of a 3-Beam Array; \( \delta = 2, \alpha = 4, \beta = 0.002, \nu = 0.002 \) \((V_{AC} = 36.83V)\).
Figure F.25: Q-Component of Frequency Response of Middle Beam of a 3-Beam Array; \( \delta = 2, \alpha = 4, \beta = 0.002, \nu = 0.002 \) (\( V_{AC} = 36.83V \)).

Figure F.26: P-Component of Frequency Response For First and Third Beams of a 3-Beam Array; \( \delta = 2, \alpha = 4, \beta = 0.002, \nu = 0.0065714 \) (\( V_{AC} = 66.76V \)).
Figure F.27: Q-Component of Frequency Response of First and Third Beams of a 3-Beam Array; $\delta = 2$, $\alpha = 4$, $\beta = 0.002$, $\nu = 0.0065714$ ($V_{AC} = 66.76V$).

Figure F.28: P-Component of Frequency Response For Middle Beam of a 3-Beam Array; $\delta = 2$, $\alpha = 4$, $\beta = 0.002$, $\nu = 0.0065714$ ($V_{AC} = 66.76V$).
Figure F.29: Q-Component of Frequency Response of Middle Beam of a 3-Beam Array; 
$\delta = 2, \alpha = 4, \beta = 0.002 (Q = 500), \nu = 0.0065714 (V_{AC} = 66.76V)$. 
Appendix G  Response Stability Analysis Graphs

In this appendix we present additional graphs of the roots of the $b_6 = 0$ equation found in Section 4.2.2. In Figure G.30, where $\beta = 0.002$ ($Q = 500$), all six roots of $b_6 = 0$ have solutions, and for particular values of $\nu$ there are four simultaneous solutions, which means that the frequency response of the system under these parameters will have two peaks. In Figure G.31 $\beta = 0.006$ ($Q = 166.7$) and only three real roots to the equation $b_6 = 0$ exist which means that only a single peak solution is possible. Figure G.32 shows that for a small number of values of $\nu$ a single peak solution can be obtained with a linear damping parameter $\beta = 0.008$ ($Q = 125$). In Figure G.33, where $\nu = 0.001$ ($V_{AC} = 26.04V$), three real roots of $b_6 = 0$ exist, but one is on the verge of disappearing. It appears that for particular values of $\beta$, a single peak solution to the system equations exists. In Figures G.34 and G.35, where $\nu = 0.006$ ($V_{AC} = 63.79V$) and $\nu = 0.008$ ($V_{AC} = 73.66V$), respectively, we see that the first five roots of $b_6 = 0$ have real (unique) solutions. Depending on the value of $\beta$ a single peak response or coexisting in-phase-out-of-phase responses can be achieved for these values of $\nu$.

Figure G.30: $\nu$ as a Function of $\Omega_{b_6=0}$; $\beta = 0.002$ ($Q = 500$).
Figure G.31: \( \nu \) as a Function of \( \Omega_{b_6}=0; \beta = 0.006 (Q = 166.7) \).
Figure G.33: $\beta$ as a Function of $\Omega_{b_0}=0$; $\nu = 0.001$ ($V_{AC} = 26.04V$).

Figure G.34: $\beta$ as a Function of $\Omega_{b_0}=0$; $\nu = 0.006$ ($V_{AC} = 63.79V$).
Figure G.35: $\beta$ as a Function of $\Omega_{b_0=0}; \nu = 0.008$ ($V_{AC} = 73.66V$).
Appendix H  Primary Resonances of First Five Modes

We present here the primary resonances for the first five modes of the single beam. Note that the ratios of the primary resonances of the different modes are very close to integers, which justifies our choice of the string mode shape.

![Diagram of primary resonances for first five modes](image)

Figure H.36: Primary Resonances of First Five Modes of Our Device.

Appendix I  Material Properties of the Sample

The only unknowns in Eq. (6.4.6) are $E$ and $\rho$. We substitute our $\tilde{\xi}$ and $f_1$ values back into Eq. (6.4.6) and obtain the ratio:

$$\left(\frac{E}{\rho}\right)_{DC_{meas}} = 4.563 \times 10^6 \quad (I-16)$$

We calculate the material properties (the elastic modulus $E$, and the density $\rho$) of our device using the law of mixtures [24] based off of the intended layer thicknesses of the beam ($Cr = 10 \text{ nm}$, $PdAu(1 : 3) = 170 \text{ nm}$) (assuming that the difference between the planned thickness, 180 nm, and the actual thickness, 165 nm, was distributed evenly) and using the material property values for the thin films of the individual metals ($E_{Au} = \ldots$)
$77.2 \text{ GPa}$, $E_{\text{Pd}} = 117 \text{ GPa}$, $E_{\text{Cr}} = 248 \text{ GPa}$, $\rho_{\text{Au}} = 19.32 \text{ g/cc}$, $\rho_{\text{Pd}} = 12.02 \text{ g/cc}$, $\rho_{\text{Cr}} = 7.19 \text{ g/cc}$) [33]. Since the height of the beam is assumed to be homogeneous, the law of mixtures is:

$$
\frac{B_{\text{Au}}MP_{\text{Au}} + B_{\text{Pd}}MP_{\text{Pd}} + B_{\text{Cr}}MP_{\text{Cr}}}{B},
$$

where $B_{\text{Au}}$, $B_{\text{Pd}}$, $B_{\text{Cr}}$ are the thicknesses of the layers of each respective metal, $MP_{\text{Au}}$, $MP_{\text{Pd}}$, $MP_{\text{Cr}}$ are the material property values of each respective metal, and $B$ is the total thickness of the beam.

This gives us a density value of $\rho = 16922.5 \text{ kg/m}^3$ and an elastic Modulus of $E = 96.086 \text{ GPa}$. (Note that since we could not find that material properties of the PdAu alloy, we treated it as if it were two separate layers with thicknesses proportional to the percentage of each metal from which the alloy is composed, namely, three parts Gold to one part Palladium.) We take the ratio of $\frac{E}{\rho}$ from the values we obtained using the law of mixtures and find that:

$$
\frac{E_{\text{LOM}}}{\rho_{\text{LOM}}} = 5.678 \times 10^6 \text{ m}^2/\text{s}^2
$$

We define the error as:

$$
\frac{\frac{E_{\text{LOM}}}{\rho_{\text{LOM}}} - \left( \frac{E}{\rho} \right)_{\text{DCmeas}}}{\frac{E_{\text{LOM}}}{\rho_{\text{LOM}}}},
$$

and find that we have a 19.64% error.

In order to correct for this error, we multiply the elastic modulus we obtained from the law of mixtures by $\sqrt{\frac{\rho_{\text{DCmeas}}}{\rho_{\text{LOM}}}}$ to obtain a new elastic modulus of $E = 86.136 \text{ GPa}$, and multiply the law of mixtures density by $\sqrt{\frac{\rho_{\text{DCmeas}}}{\rho_{\text{LOM}}}}$ to obtain a new density value of $\rho = 18,877.3 \text{ kg/m}^3$. We substitute this $E$ value and the $\xi$ value we obtained from Eq. (6.4.7) back into Eq. (6.4.1) to obtain an initial pretension of $N = 5.478 \times 10^{-5} \text{ kg/s}^2$.

Using the same process, but retaining the elastic modulus from the law of mixtures and adjusting to law of mixtures density gives us a new density of $\rho = 21057.6 \text{ kg/m}^3$ and a pretension value of $N = 5.4748 \times 10^{-5} \text{ kg/s}^2$. If we keep the density calculated from the law of mixtures, and adjust the elastic modulus, we obtain a new elastic modulus of $E = 77.217 \text{ GPa}$ and a pretension of $N = 4.908 \times 10^{-5} \text{ kg/s}^2$.

### Appendix J  Fitting the Parameters

Figure J.37 shows the results of trying to fit the data to a randomly picked parameter set. Figure J.38 shows the results of fitting the data to a parameter set consisting of a
nonlinear damping parameter that is equal to 100 times the linear damping parameter and satisfying Eq. (6.4.10).

**J-1 Fitting the Saddle Point/Maximum Response Frequency Ratio**

We calculate the ratio between the Frequencies of the saddle point and the maximum response analytically from Eqs. (4.1.21) and (4.1.30), again substituting in Eq. (6.4.9) for \( \nu \), and find that:

\[
\frac{\Omega_{saddle}}{\Omega_{max}} = \frac{2\delta(1 - 2\nu) - 3\alpha\beta + 2\nu\sqrt{9\alpha^2 + \delta^2}}{2\delta(1 - 2\nu) - 3\alpha\beta + 6\alpha\nu}
\]

\[
= \frac{2\delta(1 - \sqrt{BW^2 + \beta^2}) - 3\alpha\beta + \sqrt{BW^2 + \beta^2}\sqrt{9\alpha^2 + \delta^2}}{2\delta(1 - \sqrt{BW^2 + \beta^2}) - 3\alpha\beta + 3\alpha\sqrt{BW^2 + \beta^2}} \quad (J-20)
\]

Figures J.39-J.41 plot the square root of Eq. (6.4.10) and Eq. (J-20) as a function of \( \delta \) for a fixed \( \beta \). The black solid lines represent the measured experimental values. As can be seen in these plots, the frequency ratio can not be satisfied at all for the smaller bandwidths, and for the larger bandwidth, it does not agree with the results of the magnitude ratio.
Figure J.38: Frequency Response of A Single Beam System; $\delta = 0.0396$, $\alpha = 0.1038$, $\beta = 3.96 \times 10^{-4}$ ($Q = 2525$), $\nu = 2.26 \times 10^{-4}$ ($V_{AC} = 12.36V$).

plot.

J-2 Fitting the Maximum Response Frequency

Figures J.42-J.44 plot the square root of Eq. (6.4.10) and Eq. (4.1.21) as a function of $\delta$ for a fixed $\beta$. Again, the black solid lines represent the measured experimental values. The parameters to match the smaller measured bandwidth were chosen from Figure J.42, the parameters to match the mean bandwidth were chosen from Figure J.43, and the parameters to match the larger measured bandwidth were chosen from Figure J.44. This gives us the parameter sets $\delta = 0.040765$, $\beta = 8 \times 10^{-5}$, $\nu = 4.6086 \times 10^{-5}$, $\delta = 0.037591$, $\beta = 7.65 \times 10^{-5}$, $\nu = 4.3100 \times 10^{-5}$, and $\delta = 0.034329$, $\beta = 7.2 \times 10^{-5}$, $\nu = 3.9743 \times 10^{-5}$. These parameter pairs all have a saddle point located at $\Omega_{saddle} = 0.999957$, which is to be compared to the measured value of $\Omega_{saddle} = 0.999996$.

Note that if we substitute these parameter values into the left side of Eq. (6.4.12) we find that it is equal to 509.7 for the larger bandwidth parameter set, 491.4 for the mean bandwidth parameter set, and 476.9 for the smaller bandwidth parameter set. The right side is equal to 47.04 with our dimensions. The equality is off by a factor of ten.
Figure J.39: Saddle and Maximum Magnitude and Frequency Ratios for a Fixed Linear Damping Value ($\beta = 7 \times 10^{-5}$) as a Function of Nonlinear Damping ($\delta$).

Figure J.40: Saddle and Maximum Magnitude and Frequency Ratios for a Fixed Linear Damping Value ($\beta = 7.5 \times 10^{-5}$) as a Function of Nonlinear Damping ($\delta$).
Figure J.41: Saddle and Maximum Magnitude and Frequency Ratios for a Fixed Linear Damping Value ($\beta = 6.5 \times 10^{-5}$) as a Function of Nonlinear Damping ($\delta$).

Figure J.42: Saddle and Maximum Magnitude Ratio and Frequency of Maximum Response for a Fixed Linear Damping Value ($\beta = 7.2 \times 10^{-5}$) as a Function of Nonlinear Damping ($\delta$). Parameters Fit for $3.3675 \times 10^{-5}$ Bandwidth.
Figure J.43: Saddle and Maximum Magnitude Ratio and Frequency of Maximum Response for a Fixed Linear Damping Value ($\beta = 8 \times 10^{-5}$) as a Function of Nonlinear Damping ($\delta$). Parameters Fit for a $4.5779 \times 10^{-5}$ Bandwidth.

Figure J.44: Saddle and Maximum Magnitude Ratio and Frequency of Maximum Response for a Fixed Linear Damping Value ($\beta = 7.65 \times 10^{-5}$) as a Function of Nonlinear Damping ($\delta$). Parameters Fit for $3.9727 \times 10^{-5}$ Bandwidth.
Figure J.45: Saddle and Maximum Magnitude Ratio and Branch Off Frequency for a Fixed Linear Damping Value ($\beta = 4.706 \times 10^{-4}$) as a Function of Nonlinear Damping ($\delta$). Parameters Fit for $3.3675 \times 10^{-5}$ Bandwidth.

### J-3 Parameter Fitting for Smaller Bandwidth Estimates

Figures J.45-J.47 plot the square root of Eq. (6.4.10) and Eq. (4.1.17) as a function of $\delta$ for a fixed $\beta$. The black solid lines represent the measured experimental values. The parameters to match the smaller measured bandwidth were chosen from Figure J.45, the parameters to match the mean bandwidth were chosen from Figure J.46, and the parameters to match the larger measured bandwidth were chosen from Figure J.47. This gives us the parameter sets $\delta = 5.636 \times 10^{-3}$, $\beta = 4.706 \times 10^{-4}$ ($Q = 2125$), $\nu = 2.359 \times 10^{-4}$ ($V_{AC} = 12.65V$), $\delta = 6.707 \times 10^{-3}$, $\beta = 4.671 \times 10^{-4}$ ($Q = 2141$), $\nu = 2.344 \times 10^{-4}$ ($V_{AC} = 12.61V$), and $\delta = 7.764 \times 10^{-3}$, $\beta = 4.635 \times 10^{-4}$ ($Q = 2157$), $\nu = 2.329 \times 10^{-4}$ ($V_{AC} = 12.57V$). Note that the parallel plate estimation of the forcing magnitude, $\nu$, for a voltage $V_{AC} = 2.94$ mV, $\nu = 1.27 \times 10^{-11}$, is much smaller than that found through the parameter fit.

Substituting the parameter values we calculated for these bandwidths into the left side of Eq. (6.4.12) and the measured dimensions into the right side, we find that the left side is equal to 16.75 for the larger bandwidth parameter set, 14.36 for the mean bandwidth parameter set, and 11.98 for the smaller bandwidth parameter set. The right side is equal
Figure J.46: Saddle and Maximum Magnitude Ratio and Branch Off Frequency for a Fixed Linear Damping Value ($\beta = 4.671 \times 10^{-4}$) as a Function of Nonlinear Damping ($\delta$). Parameters Fit for a $3.9727 \times 10^{-5}$ Bandwidth.

to 47.04. For the first parameter set this is off by 64.39%, for the second is off by 69.48% and for the last it is off by 74.54%.

Substituting our parameter values into the linear transfer function, we find that for the smaller measured bandwidth $T_{11} = 3.325|x| + 0.0005041$, for the larger measured bandwidth $T_{12} = 3.069|x| - 0.0001725$, and for the mean bandwidth $T_{13} = 2.886|x| - 0.001788$. The original value can be backed out from these equations by calculating $|x| = \frac{T_{meas} - b_i}{a_i}$, where $b_i$ is the transfer function intercept and $a_i$ is the transfer function slope. This gives us the functions $|x|_1 = 0.3007T_{meas} - 1.516 \times 10^{-4}$ for the smaller bandwidth parameters, $|x|_2 = 0.3258T_{meas} + 5.622 \times 10^{-5}$ for the larger bandwidth parameters, and $|x|_3 = 0.3465T_{meas} + 6.196 \times 10^{-4}$ for the mean bandwidth parameters.

Figures J.50-J.48 show the comparison of the experimental results to the theoretical model for these three bandwidths. It is obvious from these figures that these bandwidth estimates are too small.
Figure J.47: Saddle and Maximum Magnitude Ratio and Branch Off Frequency for a Fixed Linear Damping Value ($\beta = 4.635 \times 10^{-4}$) as a Function of Nonlinear Damping ($\delta$). Parameters Fit for $4.5779 \times 10^{-5}$ Bandwidth.

Figure J.48: Comparison of Experimental Data to Theoretical Model for the parameters ($\delta = 5.636 \times 10^{-3}$, $\beta = 4.706 \times 10^{-4}$ ($Q = 2125$), $\nu = 2.359 \times 10^{-4}$ ($V_{AC} = 12.65V$)) calculated for a $3.3675 \times 10^{-5}$ Bandwidth.
Figure J.49: Comparison of Experimental Data to Theoretical Model for the parameters \( \delta = 6.707 \times 10^{-3}, \ \beta = 4.671 \times 10^{-4} \ (Q = 2141), \ \nu = 2.344 \times 10^{-4} \ (V_{AC} = 12.61V) \) calculated for a \( 3.9727 \times 10^{-5} \) Bandwidth.

Figure J.50: Comparison of Experimental Data to Theoretical Model (for the parameters \( \delta = 7.764 \times 10^{-3}, \ \beta = 4.635 \times 10^{-4} \ (Q = 2157), \ \nu = 2.329 \times 10^{-4} \ (V_{AC} = 12.57V) \) calculated for a \( 4.5779 \times 10^{-5} \) Bandwidth.
Figure J.51: Comparison of Experimental Data to Theoretical Model (for the parameters $\delta = 6.192 \times 10^{-2}$, $\beta = 3.096 \times 10^{-4}$ $(Q = 3230)$, $\nu = 1.896 \times 10^{-4}$ $(V_{AC} = 11.45V)$) calculated for a $2.189 \times 10^{-4}$ Bandwidth.

**J-4  Effect of Changing Nonlinear to Linear Damping Ratio**

In this section we present the effects of changing the $\delta$ to $\beta$ ratio from Eq. (6.4.12).

**J-5  Quadratic Transfer Function**

In order to take into account the fact that the response is branching off from the trivial solution, we derive a second transfer function which we will model by a quadratic polynomial, namely $T_{*2} = c|x|^2_q + d|x|_q + e$. In addition to the maximum and saddle point, to calculate this transfer function we will use the points where the nontrivial solution branches off from the trivial solution. At these points since $|x|_q = 0$ and the measured value $T_{2\text{zero}} = 7.475 \times 10^{-4}$, we know that $e = 7.475 \times 10^{-4}$. We are now left with a quadratic equation, $c|x|^2_q + d|x|_q + 7.475 \times 10^{-4} - T_{*2} = 0$. We solve for the inverse transfer function (adding the square root since the magnitude is a positive value) and find that:

$$|x| = -\frac{d}{2c} + \sqrt{\frac{d^2}{4c^2} + \frac{T_2 - 7.475 \times 10^{-4}}{c}}$$

(J-21)
Figure J.52: Comparison of Experimental Data to Theoretical Model (for the parameters \( \delta = 4.815 \times 10^{-2} \), \( \beta = 2.408 \times 10^{-4} \) \((Q = 4154)\), \( \nu = 1.627 \times 10^{-4} \) \((V_{AC} = 10.50V)\)) calculated for a \( 2.189 \times 10^{-4} \) Bandwidth.

Figure J.53: Comparison of Experimental Data to Theoretical Model (for the parameters \( \delta = 0.04815 \), \( \beta = 2.408 \times 10^{-4} \) \((Q = 4154)\), \( \nu = 1.896 \times 10^{-4} \) \((V_{AC} = 12.38V)\)) calculated for a \( 2.93 \times 10^{-4} \) Bandwidth.
We use Cramer’s Rule to calculate the values of the constants \( c \) and \( d \) and find that:

\[
c = \frac{T_{\text{max}} - 7.475 \times 10^{-4}}{|x|_{\text{max}}(|x|_{\text{max}} - |x|_{\text{saddle}})} - \frac{T_{\text{saddle}} - 7.475 \times 10^{-4}}{|x|_{\text{saddle}}(|x|_{\text{max}} - |x|_{\text{saddle}})} \quad (J-22)
\]

and:

\[
d = \frac{|x|_{\text{max}}(T_{\text{saddle}} - 7.475 \times 10^{-4})}{|x|_{\text{saddle}}(|x|_{\text{max}} - |x|_{\text{saddle}})} - \frac{|x|_{\text{saddle}}(T_{\text{max}} - 7.475 \times 10^{-4})}{|x|_{\text{max}}(|x|_{\text{max}} - |x|_{\text{saddle}})} \quad (J-23)
\]

Substituting the parameters for the \( 2.189 \times 10^{-4} \) bandwidth into Eqs. (J-22) and (J-23), we find that the transfer function is \( T_{*2} = 0.1520|x|_q^2 + 1.261|x|_q + 7.475 \times 10^{-4} \). In order to back out the original value from the data points we will use the inverse transfer function \( |x|_q = -4.148 + \sqrt{17.20 + 6.577T_{*2}} \). Figure J.54 shows the comparison of the theory and experiment using this transfer function.

Substituting our smaller bandwidth estimate parameters into Eqs. (J-22) and (J-23), we find that the transfer function that corresponds to the smaller measured bandwidth is \( T_{21} = 0.2954|x|_{21}^2 + 3.308|x|_{21} + 7.475 \times 10^{-4} \), the transfer function that corresponds to the larger measured bandwidth is \( T_{22} = 0.9379|x|_{22}^2 + 3.010|x|_{22} + 7.475 \times 10^{-4} \), and the transfer function that corresponds to the mean bandwidth is \( T_{23} = 2.210|x|_{23}^2 + 2.736|x|_{23} + 7.475 \times 10^{-4} \). In order to back out the original value from the data points we will use...
the inverse transfer functions $|x|_{21} = -5.599 + \sqrt{31.35 + 3.385T_{\text{meas}}}$, $|x|_{22} = -1.605 + \sqrt{2.574 + 1.066T_{\text{meas}}}$, and $|x|_{23} = -0.6189 + \sqrt{0.3827 + 0.4524T_{\text{meas}}}$.

If we use the quadratic transfer function, with the parameters from the maximum response frequency fit into Eqs. (J-22) and (J-23), we find that the transfer function that corresponds to the smaller measured bandwidth is $T_{21} = 0.2092|x|_{21}^2 + 2.823|x|_{21}$, the transfer function that corresponds to the larger measured bandwidth is $T_{22} = -0.01471|x|_{22}^2 + 3.297|x|_{22}$, and the transfer function that corresponds to the mean bandwidth is $T_{23} = 0.002470|x|_{23}^2 + 3.036|x|_{23}$. This corresponds to the inverse transfer functions $|x|_{21} = -7.878 + \sqrt{62.06 + 4.779T_{\text{meas}}}$, $|x|_{22} = 95.93 + \sqrt{9204 - 67.97T_{\text{meas}}}$, and $|x|_{23} = -614.6 + \sqrt{377800 + 404.9T_{\text{meas}}}$.
דינמיקה לא LINEARITY McCoyrovית

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שויין McCoyrov מיכֶר McCoyrov הדנה McCoyrov פרמידי

שרובט מיכֶר
דרומיקה לא לייזארית ריציבות
של מערך מכרים-קורות דנהות לא ערור פרמטרי

תוובר ע"י מחקר

לישם מילוי חלקי של הדרישות לבלת התואר
מאניסטר להדיע את הבנדת מכונת

שובה פורטינן

ה🐙 לשנאות משכונור — מכון טכנולוגיה ליוואלא
תפוזו תשש"ט

וולי 2009 החיפה
המחזור הע butterknife פרופסור חור גורליב בבית הספר להנדסת מכונות פקולטה.

הכור אמיל בוקס ומזכירה לחתמת שמות.

בגרות בבית הספר לי לותר והנומיה להנחייה ולעניקהbows במאמר המחזור.

אני מודה לפרופסור גוטליב על העניקה והנחייה.

בגרות לי לותר מאמינים שהנחיית ועניקה bows במאמר המחזור.

אני מודה לדרור的应用 והנחייה על העניקה והנחייה.

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אני מודה בגרות לי לותר והנומיה להנחייה ועניקה bows במאמר המחזור.

לבסוף, אני מודה לך לותר והנומיה להנחייה ועניקה bows במאמר המחזור.

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6.2. המאמז

6.3. תוצאות

6.3.1. קורא על שערכו המאמז

6.3.2. קורא המאמז של שבע קורא

6.4. לשטח

6.4.1. קורא לשטח

6.4.2. קורא לשטח של שבע קורא

6.4.3. קורא לשטח של שבע קורא

6.5. דיון

7. מסכום

7.1. סיכום

7.2. מסקנות

7.3. מחקר להמשך

8. מקורות

A. נוספות

B. נוספות

C. נוספות

D. נוספות

E. נוספות

F. נוספות

G. נוספות

H. נוספות

I. נוספות

J. נוספות

J.1. נוספות

J.2. נוספות

J.3. נוספות

J.4. נוספות

J.5. נוספות
תקציר מודרבת

ミクリ- ハモノデヨ (ロゴソニヨ) トヨノモノ キゴニゴノ ものニシノ 茉ンシノ モデルシト ボゴト ハバト ハバト

 الدكتور י. בינוני עלה באומד על לי כיוון הרצוגโปรאא על ידי המ�� של מתור י. רמיה.

 החלוף חלום (מנ). כיוון הרצוגโปรאא על ידי השראת ומאמר הרצוגпроוי ששל מתור י. רמיה

 חפץ אלחוטי שאל מתור הופך המחליפה מחליפה את המחליפה את הפתח את הפתח את הפתח את הפתח את הפתח את הפתח את הפתח את הפתח את הפתח את הפתח את הפתח את הפתח את הפתח את הפתח את הפתח את הפתח את הפתח את הפתח את הפתח את הפתח את הפתח את הפתח את הפתח את הפתח את הפתח את הפתח את הפתח את הפתח את הפתח את הפתח את הפתח את הפתח את הפתח את הפתח את הפתח את הפתח את הפתח את הפתח את הפתח את הפתח את הפתח את הפתח את הפתח את הפתח את הפתח את הפתח את הפתח את הפתח את הפתח את הפתח את הפתח את הפ parachאתיור לי. למתקדיץ המחליפה את המחליפה את הפתח את הפתח את הפתח את הפusaha את הפusaha את הפusaha את הפusaha את הפusaha את הפusaha את הפusaha את הפusaha את הפusaha את הפusaha את הפusaha את הפusaha את הפusaha את הפusaha את הפusaha את הפusaha את הפusaha את הפusaha את הפusaha את הפusaha את הפusaha את הפusaha את הפusaha את הפusaha את הפusaha את הפusaha את הפusaha את הפusaha את הפusaha את הפusaha את הפusaha את הפusaha את הפusaha את הפusaha את הפusaha את הפusaha את הפusaha את הפusaha את הפusaha את הפusaha את הפusaha את הפusaha את הפusaha את הפusaha את הפusaha את הפusaha את הפusaha את הפusaha את הפusaha את הפusaha את הפusaha את הפusaha את הפusaha את הפusaha את הפusaha את הפusaha את הפusaha את הפusaha את הפusaha את הפusaha את הפusaha את הפusaha את הפusaha את הפusaha את הפusaha את הפusaha את הפusaha את הפusaha את הפusaha את הפusaha את הפusaha את הפusaha את הפusaha את הפusaha את הפusaha את הפusaha את הפusaha את הפusaha את הפusaha את הפusaha את הפusaha את הפusaha את הפusaha את הפusaha את הפusaha את הפusaha את הפusaha את הפusaha את הפusaha את הפusaha את הפusaha את הפusaha את הפusaha את הפשרה את הפusaha את הפיעה את הפיאה את הפיאה את הפיאה את הפיאה את הפיאה את הפיאה את הפיאה את הפיאה את הפיאה את הפיאה את הפיאה את הפיאה את הפיאה את הפיאה את הפיאה את הפיאה את הפיאה את הפיאה את הפיאה את הפיאה את הפיאה את הפיאה את הפיאה את הפיאה את הפיאה את הפיאה את הפיאה את הפיאה את הפיאה את הפיאה את הפיאה את הפיאה את הפיאה את הפיאה את הפיאה את הפיאה את הפיאה את הפיאה את הפיאה את הפיאה את הפיאה את הפיאה את הפיאה את הפיאה את הפיאה את הפיאה את הפיאה את הפיאה את הפיאה את הפיאה את הפיאה את הפיאה את הפיאה את הפיאה את הפיאה את הפיאה את הפיאה את הפיאה את הפיאה את הפיאה את הפיאה את הפיאה את הפיאה את הפיאה את הפיאה את הפיאה את הפיאה את הפיאה את הפיאה את הפיאה את הפיאה את הפיאה את הפיאה את הפיאה את הפיאה את הפיאה את הפיאה את הפיאה את הפיאה את הפיאה את הפיאה את הפיאה את הפיאה את הפיאה את הפיאה את הפיאה את הפיאה את הפיאה את הפיאה את הפיאה את הפיאה את הפיאה את הפיאה את הפיאה את הפיאה את הפיאה את הפיאה את הפיאה את הפיאה את הפיאה את הפיאה את הפיאה את הפיאה את הפיאה את הפיאה את הפיאה את הפיאה את הפיאה את הפיאה את הפיאה את הפיאה את הפיאה את הפיאה את הפיאה את הפיאה את הפיאה את הפיאה את הפיאה את הפיאה את הפיאה את הפיאה את הפיאה את הפיאה את הפיאה את הפיאה את הפיאה את הפיאה את הפיאה את הפיאה את הפיאה את הפיאה את הפיאה את הפיאה את הפיאה את הפיאה את הפיאה את הפיאה את הפיאה את הפיאה את הפיאה את הפיאה את הפיאה את הפיאה את הפיאה את הפיאה את הפיאה את הפיאה את הפיאה את הפיאה את הפיאה את הפיאה את הפיאה את הפיאה את הפיאה את הפיאה את הפיאה את הפיאה את הפיאה את הפיאה את הפיאה את הפיאה את הפיאה את הפיאה את הפיאה את הפיאה את הפיאה את הפיאה את הפיאה את הפיאה את הפיאה את הפיאה את הפיאה את הפיאה את הפיאה את הפיאה את הפיאה את הפיאה את הפיאה את הפיאה את הפיאה את הפיאה את הפיאה את הפיאה את הפיאה את הפיאה את הפיאה את הפיאה את הפיאה את הפיאה את הפיאה את הפיאה את הפיאה את הפיאה את הפיאה את הפיאה את הפיאה את הפיאה את הפיאה את הפיאה את הפיאה את הפיאה את הפיאה את הפיאה את הפיאה את הפיאה את הפיאה את הפיאה את הפיאה את הפיאה את הפיאה את הפיאה את הפיאה את הפיאה את הפיאה את הפיאה את הפיאה את הפיאה את הפיאה את הפיאה את הפיאה את הפיאה את הפיאה את הפיאה את הפיאה את הפיאה את הפיאה את הפיאה את הפיאה את הפיאה את הפיאה את הפיאה את הפיאה את הפיאה את הפיאה את הפיאה את הפיאה את הפיאה את הפיאה את הפיאה את הפיאה את הפיאה את הפיאה את הפיאה את הפיאה את הפיאה את הפיאה את הפיאה את הפיאה את הפיאה את הפיאה את הפיאה את הפיאה את הפיאה את הפיאה את הפיאה את הפיאה את הפיאה את הפיאה את הפיאה את הפיאה את הפיאה את הפיאה את הפיאה את הפיאה את הפיאה את הפיאה את הפיאה את הפיאה את הפיאה את הפיאה את הפיאה את הפיאה את הפיאה את הפיאה את הפיאה את הפיאה את הפיאה את הפיאה את הפיאה את הפיאה את הפיאה את הפיאה את הפיאה את הפיאה את הפיאה את הפיאה את הפיאה את הפיאה את הפיאה את הפיאה את הפיאה את הפיאה את הפיאה את הפיאה את הפיאה את הפיאה את הפיאה את הפיאה את הפיאה את הפיאה את הפיאה את הפיאה את הפיאה את הפיאה את הפיאה את הפיאה את הפיאה את הפיאה את הפיאה את הפיאה את הפיאה את הפיאה את הפיאה את הפיאה את הפיאה את הפיאה את הפיאה את הפיאה את הפיאה את הפיאה את הפיאה את הפיאה את הפיא
The research team conducted a study on the free silicon and investigated the five parameters with a chemical material. They wrote that the weak part of the equation is the linear dependency between the influence of the exit and the stability of the system. The exit of the IC (1:1:1) is centrifugal, but it has partial shortcomings. According to the selected analytical parameters, and the partial model, the first stage of the research was solved. The researchers conducted a comparison between the solutions and found that the second model matches the solutions for the second model. They also chose the scale and described the solution of the eigenvalues, the resonance of each parameter, and the eigenvalue of the system. They further explained that the mathematical equation for the system is a differential equation that can be solved. The solution of the eigenvalues was written in the equation. They concluded that the research was conducted using silicon wafer (PMMA).
לאחר.Sn, בבעזרת ניסיון שלידיה החומרים, שלח הידברות עם התוצאות שתו. והן המשך ב_question. התוצאות של הקווים של הקווים של הקוונטים של הקווים של הקוונטים של הקווים של הקווים של הקווים של הקוונטים של הקווים של הקוונטים של הקווים של הקוונטים של הקוונטים של הקווים של הקוונטים של הקווים של הקוונטים של הקוונטים של הקווים של הקוונטים של הקוונטים של הקוונטים של הקוונטים של הקוונטים של הקוונטים של הקווים של הקוונטים של הקוונטים של הקוונטים של הקוונטים של הקוונטים של הקוונטים של הקוונטים של הקוונטים של הקוונטים של הקוונטים של הקוונטים של הקוונטים של הקוונטים של הקוונטים של הקוונטים של הקוונטים של הקוונטים של הקוונטים של הקוונטים של הקוונטים של הקוונטים של הקוונטים של הקוונטים של הקוונטים של הקוונטים של הקוונטים של הקוונטים של הקוונטים של הקוונטים של הקוונטים של הקוונטים של הקוונטים של הקוונטים של הקוונטים של הקוונטים של הקוונטים של הקוונטים של הקוונטים של הקוונטים של הקוונטים של הקוונטים של הקוונטים של הקוונטים של הקוונטים של הקוונטים של הקוונטים של הקוונטים של הקוונטים של הקוונטים של הקוונטים של הקוונטים של הקוונטים של הקוונטים של הקוונטים של הקוונטים של הקוונטים של הקוונטים של הקוונטים של הקוונטים של הקוונטים של הקוונטים של הקוונטים של הקוונטים של הקוונטים של הקוונטים של הקוונטים של הקוונטים של הקוונטים של הקוונטים של הקוונטים של הקוונטים של הקוונטים של הקוונטים של הקוונטים של הקוונטים של הקוונטים של הקוונטים של הקוונטים של הקוונטים של הקוונטים של הקוונטים של הקוונטים של הקוונטים של הקוונטים של הקוונטים של הקוונטים של linea.