Off-resonance coupling between a cavity mode and an ensemble of driven spins

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We study the interaction between a superconducting cavity and a spin ensemble. The response of a cavity mode is monitored while simultaneously the spins are driven at a frequency close to their Larmor frequency, which is tuned to a value much higher than the cavity resonance. We experimentally find that the effective damping rate of the cavity mode is shifted by the driven spins. The measured shift in the damping rate is attributed to the retarded response of the cavity mode to the driven spins. The experimental results are compared with theoretical predictions and fair agreement is found.

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I. INTRODUCTION

Cavity quantum electrodynamics (CQED) [1] is the study of the interaction between matter and photons confined in a cavity. In the Jaynes-Cummings model [2], the matter is described using the two-level approximation and only a single-cavity mode is taken into account. The interaction has a relatively large effect on the cavity mode response when the ratio \( E/\hbar \omega_a \) between the energy gap \( E \) separating the two levels and the cavity mode photon energy \( \hbar \omega_a \) is tuned close to unity. Recently, it was experimentally found that the cavity response exhibits higher-order resonances in the nonlinear regime when the ratio \( E/\hbar \omega_a \) is tuned close to an integer value larger than unity [3].

In the current study, we explore the case where \( E/\hbar \omega_a \gg 1 \) [4]. This is done by investigating the interaction between an ensemble of spins and a superconducting cavity mode [5–7]. The energy separation between the spin energy eigenstates, which is given by \( E = \hbar \omega_a \), where \( \omega_a \) is the Larmor frequency, is tuned to a value much higher than the cavity mode photon energy \( \hbar \omega_a \). For this case, the CQED interaction is expected to be negligibly small in the regime of weak driving. On the other hand, with an intense driving at an angular frequency close to \( \omega_a \), we observe a significant change in the cavity mode response.

In the current experiment, the cavity mode effective damping rate is measured as a function of the spin driving amplitude and detuning frequency. The observed shift in the effective damping rate is attributed to the retarded response of the cavity mode to the driven spins. Related effects of Sisyphus cooling, amplification, lasing, and self-excited oscillation have been theoretically predicted in other systems having a similar retarded response [8–12].

II. EXPERIMENT

Significant change in the response of the measured cavity mode of angular frequency \( \omega_a \) is possible only when intense driving is applied to the spins. In order to allow sufficiently strong driving, the spin ensemble is coupled to an additional cavity mode having angular frequency \( \omega_b \gg \omega_a \). When the Larmor frequency \( \omega_a \) is tuned to a value close to \( \omega_b \), the additional cavity mode allows enhancing the spin driving amplitude.

A sketch of the device is seen in Fig. 1. It is made of two sapphire wafers and a high-resistivity silicon wafer that are attached together to form a dual band resonator. A radio-frequency resonator of angular frequency \( \omega_b \) is constructed by integrating an inductor in the shape of the Greek letter \( \Omega \) [13] made on the bottom sapphire wafer, and two capacitors in series, which are formed between the two sapphire wafers. A square hole is made in the upper sapphire wafer in order to allow insertion of the silicon wafer, which carries a spiral-shaped microwave resonator having angular frequency \( \omega_a \) [14,15].

Both of the resonators are designed to be efficiently coupled to the spin ensemble of diphenylypicrylhydrazyl (DPPH) powder, placed between them. This radical, which contains three benzene rings, has a single unpaired electron, which gives rise to a Landé g-factor of 2.0036 [16,17]. A sketch of the experimental setup is seen in Fig. 2. A loop antenna is employed for delivering input and output signals to both resonators.

The measured reflectivity near the electron spin resonance (ESR) of the omega and spiral resonators is seen in Figs. 3(a) and 3(b), respectively [18]. Fitting the data with theory (e.g., Eq. (4) of Ref. [3]) allows extracting the value of the coupling coefficient \( g_a \). A sketch of the experiment is seen in Fig. 1. A loop antenna is employed for delivering input and output signals to both resonators.

The line shape of the measured cavity reflectivity vs frequency curves. The change in the damping rate, \( -\text{Im} \Gamma_a \), is seen in the color-coded plots of Fig. 4 as a function of the Larmor frequency \( \omega_a \) and the spin driving angular frequency \( \omega_b \).

III. THEORY

To account for the experimental findings, two possible contributions to \( \Gamma_a \), which is expressed as \( \Gamma_a = \Gamma_{ab} + \Gamma_{ab} \), have been theoretically estimated. While \( \Gamma_{ab} \) represents the shift induced by the coupling to the driven spins, the \( \Gamma_{ab} \) contribution originates from the coupling to the driven spiral mode.

\[ \text{(4)} \]
The device is made of two 40×40×0.5 mm sapphire wafers carrying the radio-frequency omega resonator, and a 5×5×0.5 mm silicon wafer carrying the microwave spiral resonator. The DPPH powder is placed between the omega inductor and the spiral. The three wafers are vertically shifted in the sketch for clarity. In the assembled device, both the top sapphire wafer and the silicon wafer are placed directly on top of the bottom sapphire wafer. The three wafers and a loop antenna are assembled together inside a package made of high-conductivity oxygen-free copper. Both omega spiral resonators are made by dc-magnetron sputtering of a 200-nm-thick niobium layer. The radius of the omega inductor is 500 μm and the linewidth is 40 μm. The spiral dimensions are inner radius 500 μm, outer radius 580 μm, linewidth 10 μm, and number of turns 4. The measured frequency of the omega (spiral) resonator is ω₀/2π = 0.173 GHz (ω₀/2π = 2.00 GHz), whereas the value obtained from numerically simulating the structure is 0.176 GHz (2.07 GHz).

FIG. 2. The experimental setup. A power combiner (PC) is employed for combining the injected signals of a signal generator (SG) and a network analyzer (NA). The combined injected signal is transmitted through an amplifier (A) and a coupler (C), and feeds the loop antenna (LA), which is positioned above the device under study (DUS). The back-reflected signal is split by a power splitter (PS) and measured by both a NA and a spectrum analyzer (SA).

A magnetic field having two mutually orthogonal components, i.e., a static component and an alternating one at an angular frequency ωp, is applied to the spin ensemble. The amplitude of the static (alternating) component is γ⁻¹ωL (γ⁻¹ω1), where γL is the electron spin gyromagnetic ratio. The frequency shift ΔωL is found to be given by [see Appendix A and Eq. (A46)]

$$\Delta \omega_L = \frac{\eta aL}{ω_1} \left( i \frac{ω_p}{ω_1} - 1 \right) - i \left( \frac{ω_p}{ω_1} - 1 \right),$$

where ΔωL = ωp − ωL is the detuning, γ₁ (γ₂) is the longitudinal (transverse) spin relaxation rate, p0 is the spin polarization in thermal equilibrium [see Eq. (A20)], ωR = \sqrt{4ω_1^2 + Δ\omega_L^2} is the Rabi frequency of the driven spins, and η is given by η = (2γ₂/γ₁)[2ω₁(1 − γ₁/γ₂)/ω₂ − 1] [see Eq. (A47)]. Note that Eq. (1) is obtained by assuming that |Δ\omega_L| ≪ ω₁, γ₂ ≪ ω₁, and γ₁, γ₂ ≪ ω₁.

The real part of ΔωL is the cavity mode angular frequency change that is induced by the coupling to the driven spins, whereas the imaginary part is −γΔωL, where γΔωL denotes the change in cavity mode damping rate. The dependence of the normalized change in damping rate γΔωL/ω₁ on the normalized detuning ΔωL/ω₁ and the normalized driving amplitude ω₁/ω₉ is shown in Fig. 5(a). When the driving is red detuned, i.e., when ΔωL is negative, the change in damping rate γΔωL is positive and, consequently, mode cooling is expected to occur [19]. The opposite behavior occurs with blue detuning, i.e., when ΔωL is positive.
The Larmor frequency 
\( \omega \) of the spins can be explained by noticing that along the dotted line, i.e., Eq. (1) vanishes and, consequently, \( |\gamma_a| \) reaches a maximum. The largest change in the damping rate, which is denoted by \( \gamma_{ad, \text{max}} \), can be evaluated by analyzing the expression given by Eq. (1). In the absence of spin dephasing, i.e., when \( \gamma_1/\gamma_2 = 2 \), it is found that the largest change, which is given by \( \gamma_{ad, \text{max}} \approx 0.437 \times g_0^2 p_0/\gamma_2 \), occurs at the points \( (\Delta_{pl}/\omega, \omega_1/\omega_2) \approx (\pm 0.527, 0.425) \), which are labeled by crosses in Fig. 5(a). In the current experiment, however, these points are not accessible since \( \omega_1 \ll \omega_a \).

The underlying mechanism responsible for the change in the effective cavity mode damping rate is similar to a related mechanism occurring in optomechanical cavities [19]. The coupling to the spins gives rise to a forcing term acting on the cavity mode, which is proportional to the spin polarization \( p_z \) [see Eq. (A11)]. On the other hand, the same coupling effectively shifts the Larmor frequency of the spins [see Eq. (A12)] and, consequently, the effective spin driving detuning \( \Delta_{pl, \text{eff}} = \Delta_{pl} + g_0 x_a \) becomes dependent on the cavity mode amplitude \( x_a \) [see Eq. (A25)].

For any fixed value of the cavity mode amplitude \( x_a \), the spin polarization \( p_z \) in the steady state, which is denoted by \( p_{z,0} \), can be calculated using Eq. (A43) below. The dependence of \( p_{z,0} \) on \( x_a \) is demonstrated by the solid black line in Fig. 5(b) for the case of blue-detuned spin driving. Consider first the adiabatic limit, for which it is assumed that \( \omega_a \ll \gamma_1, \gamma_2 \). For this case, the dynamics of the cavity mode is assumed to be relatively slow and, consequently, the spin polarization \( p_z \) is expected to remain very close to the steady-state value given by \( p_{z,0} \), i.e., to adiabatically follow the \( x_a \)-dependent instantaneous steady-state value. Therefore, no change in the cavity mode damping rate is expected in the adiabatic limit.

Large deviation between the momentary polarization \( p_z \) and the steady-state value \( p_{z,0} \) is possible in the nonadiabatic case, for which the response of the spins to the time evolution of the cavity mode becomes retarded. The closed curve in Fig. 5(b) represents the periodic time evolution of \( p_z \) for the case where the cavity mode oscillates at a fixed amplitude at its resonance frequency around the point \( x_a = 0 \). Since \( p_z \) is proportional to the force acting on the cavity mode, the area colored in gray in Fig. 5(b) is proportional to the net work done on the cavity mode per cycle. While the area is positive for the case of blue detuning, which is the case demonstrated by Fig. 5(b), red detuning gives rise to negative values, i.e., to energy flowing away from the cavity mode. These effects of energy flow between the cavity mode and the driven spins give rise to the above-discussed change in the effective cavity mode damping rate.

The frequency shift due to the driven spiral mode is attributed to an intermode coupling term in the Hamiltonian of the coupled system, which is assumed to be given by 
\[ K (A_a + A_b^\dagger) (A_b + A_a^\dagger) ^2 \], where \( A_a \) (\( A_b \)) is an annihilation.
operator of the omega (spiral) resonator, and $K$ is the intermode coupling coefficient [see Eq. (B1)]. The contribution $\Upsilon_{ab}$ is found to be given by [see Appendix B and Eqs. (B19) and (B30)]

$$\Upsilon_{ab} = \frac{4K^2|F_d|^2}{\omega_D^2 + \gamma_b^2} \times \left\{ \left[ \frac{\omega_D - \omega_b}{\gamma_b} - 1 \right] \left[ \frac{\omega_D + \omega_b}{\gamma_b} - 1 \right] \right\} + \frac{1 + \frac{2\gamma_b}{\omega_b}}{\omega_b^2},$$

(2)

where $F_d$ and $\omega_D$ are the amplitude and angular frequency detuning, respectively, of the spiral mode driving, $\omega_b$ and $\gamma_b$ are the spiral mode angular frequency and damping rate, respectively, and $\omega_s = 2\omega_b - \omega_d$. Note that when $\gamma_b \ll \omega_b$ and $\gamma_b \ll \omega_s$, the first term in the second row of Eq. (2) becomes negligibly small provided that $|\omega_D| \ll \omega_b^2/\omega_s$.

IV. DISCUSSION

As can be seen from the comparison between Figs. 4(a) and 4(b), fair agreement is obtained between data and theory. Reasonable agreement cannot be obtained unless both contributions $\Upsilon_{al}$ [Eq. (1)] and $\Upsilon_{ab}$ [Eq. (2)] are taken into account. The contribution of $\Upsilon_{ab}$ is dominated by the second term in the second row of Eq. (2).

Our results demonstrate the ability to modify the effective damping rate of a cavity mode by driving spins that are coupled to the mode. Red-detuned driving provides a positive contribution to the damping rate, whereas negative contribution can be obtained by blue-detuned driving. For the former case, this effect can be utilized for cooling down a cavity mode, while the latter case of blue detuning may allow the self-excitation of oscillation. Operating close to the threshold of self-excited oscillation, i.e., close to the point where the total effective damping vanishes, may be useful for some sensing applications since the system is expected to become highly responsive to external perturbations near the threshold.

As was shown above, relatively large change in the damping rate can be induced provided that the Rabi frequency $\omega_D$ of the driven spins becomes comparable to the cavity mode frequency $\omega_s$ (see Fig. 5). Unfortunately, this region is inaccessible with the devices that have been investigated in the current experiment. However, in other CQED systems, the condition $\omega_D \approx \omega_s$ can be more easily satisfied. For example, with superconducting CQED systems, both strong [20–23] and ultrastrong [24,25] coupling is possible. This, together with the ability to drive a Josephson qubit with Rabi frequencies high in the radio-frequency band, may allow satisfying the condition $\omega_D \approx \omega_s$ with a strongly coupled cavity mode. As was shown above, a large change in cavity mode damping rate, of the order of $g_s^2|p_0|/\gamma_2$, is possible provided that the region where $\omega_D \approx \omega_s$ becomes accessible. For a typical superconducting CQED system, the damping rate of a decoupled cavity mode is far smaller than $g_s^2|p_0|/\gamma_2$, and thus reaching this region may allow efficient cooling down cavity modes by off-resonance qubit driving.

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APPENDIX A: COUPLING TO DRIVEN SPINS

Consider an ensemble of spin-1/2 particles coupled to a cavity mode. The ensemble is characterized by a longitudinal (spin-lattice) relaxation rate $\gamma_1$ and by a transverse (spin-spin) relaxation rate $\gamma_2$. An external magnetic field is applied, having a component alternating with angular frequency $\omega_b$, and an orthogonal static component. The amplitude of the alternating (static) component is $\gamma_b^{-1}\omega_1 (\gamma_b^{-1}\omega_\perp)$, where $\gamma_b = 2\pi \times 28.03 \text{ GHz} \times 1^{-1}$ is the electron spin gyromagnetic ratio. It is assumed that driving is applied close to the electron spin resonance, i.e., $|\Delta_{ab}| \ll \omega_b$, where $\Delta_{ab} = \omega_p - \omega_b$ is the detuning. The cavity mode is characterized by an angular frequency $\omega_s$ and a damping rate $\gamma_s$. The coupling between the cavity mode and the spin ensemble is characterized by a longitudinal coupling coefficient $g_s$.

1. Equations of motion

The Hamiltonian of the closed system is taken to be given by

$$\hat{h}^{-1}\hat{H}_{al} = \omega_a \left( A_1^\dagger A_2 + \frac{1}{2} \right) + \frac{\omega_l}{2} \hat{\Sigma}_z$$

$$+ \omega_1 (e^{-i\omega_1 t}\hat{\Sigma}_+ + e^{i\omega_1 t}\hat{\Sigma}_-) - g_s (A_1^\dagger A_2^\dagger + A_2 A_1^\dagger) \hat{\Sigma}_z,$$

(A1)

where $\omega_a$ is the cavity mode angular frequency, $A_1$ is a cavity annihilation operator, and $\hat{\Sigma}_+$ and $\hat{\Sigma}_-$ are spin operators. The Heisenberg equations of motion are generated according to

$$\frac{dO}{dt} = -i[O, \hat{h}^{-1}\hat{H}_{al}],$$

(A2)

where $O$ is an operator. Using the commutation relations

$$[A_3, A_3^\dagger] = 1,$$

(A3)

$$[\hat{\Sigma}_+, \hat{\Sigma}_+] = 2\hat{\Sigma}_+,$$

(A4)

$$[\hat{\Sigma}_-, \hat{\Sigma}_-] = -2\hat{\Sigma}_-,$$

(A5)

$$[\hat{\Sigma}_+, \hat{\Sigma}_-] = \hat{\Sigma}_z,$$

(A6)

one obtains

$$\frac{dA_2}{dt} + i\omega_a A_2 - ig_s \hat{\Sigma}_z = 0,$$

(A7)

$$\frac{d\hat{\Sigma}_+}{dt} + i\Omega_a \hat{\Sigma}_+ + i\omega_1 e^{i\omega_1 t} \hat{\Sigma}_z = 0,$$

(A8)

and

$$\frac{d\hat{\Sigma}_-}{dt} + 2i\omega_1 (\hat{\Sigma}_- e^{-i\omega_1 t} - \hat{\Sigma}_+ e^{i\omega_1 t}) = 0,$$

(A9)
where

$$\Omega_L = \omega_L - 2g_a(A_a + A_1^L).$$  \hspace{1cm} (A10)

In the next step, damping is introduced and the resultant equations for the operators $A_\lambda$, $\Sigma_+$, and $\Sigma_c$ are thermally averaged. This procedure leads to

$$\frac{da}{dt} + \Theta_a = 0,$$  \hspace{1cm} (A11)
$$\frac{dp_+}{dt} + \Theta_+ = 0,$$  \hspace{1cm} (A12)
$$\frac{dp_z}{dt} + \Theta_z = 0,$$  \hspace{1cm} (A13)

where

$$a = \langle A_a \rangle,$$  \hspace{1cm} (A14)
$$p_+ = e^{-i\omega_L t} \langle \Sigma_+ \rangle,$$  \hspace{1cm} (A15)
$$p_c = \langle \Sigma_c \rangle = p_z,$$  \hspace{1cm} (A16)

triangle brackets denote thermal averaging, the functions $\Theta_a$, $\Theta_+$, and $\Theta_z$ are given by

$$\Theta_a = \lambda_a a - ig_a p_z,$$  \hspace{1cm} (A17)
$$\Theta_+ = (i \Delta_{pl} + \gamma_2)p_+ + i\omega_1 p_z + 2ig_a(a + a^*)p_+,$$  \hspace{1cm} (A18)
$$\Theta_z = \gamma_1(p_z - p_0) + 2i\omega_1(p_+ - p_+^*),$$  \hspace{1cm} (A19)

the cavity eigenvalue $\lambda_a$ is given by $\lambda_a = i\omega_a + \gamma_a$, the coefficient

$$p_0 = - \tanh \left( \frac{\hbar \omega_a}{2k_BT} \right)$$  \hspace{1cm} (A20)

is the value of $p_z$ in thermal equilibrium in the absence of both driving and coupling, $k_B$ is the Boltzmann’s constant, and $T$ is the temperature.

2. The cavity eigenvalue

The $5 \times 5$ Jacobian matrix

$$J = \frac{\delta(\Theta_a, \Theta_+, \Theta_+, \Theta_c)}{\delta(a, a^*, p_+, p_+^*, p_z)}$$  \hspace{1cm} (A21)

can be expressed as $J = J_0 + g_a V$, where the matrix $J_0$ in a block form is given by

$$J_0 = \begin{pmatrix}
\lambda_a & 0 & 0 & 0 & -i \\
0 & \lambda_m^* & 0 & 0 & i \\
0 & 0 & -i\omega_1 & 0 & 0 \\
0 & 0 & 0 & -2i\omega_1 & \gamma_1
\end{pmatrix}$$  \hspace{1cm} (A22)

the block $J_L$ is given by

$$J_L = \begin{pmatrix}
i\Delta_{pl} + \gamma_2 & 0 & i\omega_1 \\
0 & -i\Delta_{pl} + \gamma_2 & -i\omega_1 \\
2i\omega_1 & -2i\omega_1 & \gamma_1
\end{pmatrix}.$$  \hspace{1cm} (A23)

the matrix $V$ is given by

$$V = \begin{pmatrix}
0 & 0 & 0 & 0 & -i \\
0 & 0 & 0 & 0 & i \\
-2ip_+ & -2ip_+^* & 0 & -i\omega_a & 0 \\
0 & 0 & 0 & 0 & 0
\end{pmatrix},$$  \hspace{1cm} (A24)

and

$$x_a = 2(a + a^*).$$  \hspace{1cm} (A25)

Let $\lambda_1, \lambda_2, \ldots, \lambda_5$ be the five eigenvalues of $J = J_0 + g_a V$. In the limit $g_a \to 0$, i.e., when the cavity mode is decoupled from the spins, it is assumed that $\lambda_1 \to \lambda_a$. When $g_a$ is sufficiently small, the eigenvalue $\lambda_1$, which henceforth is referred to as the cavity eigenvalue, can be calculated using perturbation theory. For the case of high-quality factor (i.e., the case where $\gamma_a \ll \omega_a$), $\lambda_1$ is found to be given to second order in $g_a$ by

$$\lambda_1 = i\omega_a + \gamma_a + g_a V - g_a^2 [V R(\omega_a) V]_{11} + O(g_a^3),$$  \hspace{1cm} (A26)

where the $5 \times 5$ matrix $R(\omega)$ in a block form is given by

$$R(\omega) = \begin{pmatrix}
0 & 0 & 0 & 0 \\
0 & 0 & 0 & \chi_L(\omega') \\
0 & \chi_L(\omega') & 0 & 0 \\
\chi_L(\omega') & 0 & 0 & 0
\end{pmatrix},$$  \hspace{1cm} (A27)

where the $3 \times 3$ spin susceptibility matrix $\chi_L(\omega')$ is given by

$$\chi_L(\omega') = (J_L - i\omega')^{-1}.$$  \hspace{1cm} (A28)

With the help of Eq. (A24), one finds that

$$\lambda_1 = i\omega_a + \gamma_a + \Lambda_1 + O(g_a^3),$$  \hspace{1cm} (A29)

where

$$\Lambda_1 = 2g_a^2 [p_+^* (\lambda_1(\omega_a))_{12} - p_+[\lambda_1(\omega_a)]_{13}].$$  \hspace{1cm} (A30)

The following holds [see Eq. (A23)]:

$$\chi_L(\omega_a) = \frac{1}{D_L} \begin{pmatrix}
D_2 D_3 + 2\omega_1^2 & 2\omega_1^2 & -i\omega_1 D_2 \\
2\omega_1^2 & D_1 D_3 + 2\omega_1^2 & i\omega_1 D_1 \\
-2i\omega_1 D_2 & 2i\omega_1 D_1 & D_1 D_3
\end{pmatrix},$$  \hspace{1cm} (A31)

where

$$D_1 = i\Delta_{pl} + \gamma_2 - i\omega_a,$$  \hspace{1cm} (A32)
$$D_2 = -i\Delta_{pl} + \gamma_2 - i\omega_a,$$  \hspace{1cm} (A33)
$$D_3 = \gamma_1 - i\omega_a,$$  \hspace{1cm} (A34)
$$D_L = D_1 D_2 D_3 + 2\omega_1^2 (D_1 D_3 + D_2).$$  \hspace{1cm} (A35)

The determinant $D_L$ can be expressed as [see Eq. (A35)]

$$\frac{D_L}{\omega_a^3} = \omega_a \left( \frac{\Delta_{pl}^2 - \omega_{DR}^2}{\omega_a^2} \right) - i \left( \frac{\Delta_{pl}^2 - \omega_{DR}^2}{\omega_a^2} \right),$$  \hspace{1cm} (A36)

where

$$\frac{\omega_{DR}}{\omega_a} = \sqrt{1 + \frac{2\gamma_2}{\gamma_1} \frac{1 - \frac{2\omega_1^2}{\omega_a^2} - \frac{\gamma_1^2}{\omega_a^2}}{1 - \frac{\gamma_1^2}{\omega_a^2}}}.$$  \hspace{1cm} (A37)
and
\[ \frac{\omega_{\text{fd}}}{\omega_a} = \sqrt{1 - \frac{4\omega_1^2}{\omega_a^2} - \frac{(2\gamma_1 + \gamma_2)\gamma_2}{\omega_a^2}}. \tag{A38} \]

Using these notations, Eq. (A30) becomes
\[ \frac{\Lambda_1}{\omega_a} = \frac{8g_1^2\omega_1}{\omega_a^3} \frac{i\rho''(\omega_1 + \omega_0) + \rho'(1 + i\frac{\rho}{\omega_1})}{\frac{\rho}{\omega_1}}, \tag{A39} \]

where \( \rho' \) (\( \rho'' \)) is the real (imaginary) part of \( \rho \), i.e.,
\[ \rho' = \frac{\rho + \rho''}{2}, \tag{A40} \]
\[ \rho'' = \frac{\rho - \rho'}{2i}. \tag{A41} \]

To second order in \( g_a \), the term \( \Lambda_1 \) [see Eq. (A39)] can be calculated by evaluating the fixed point value of \( p_+ \) to zeroth order in \( g_a \), which is done by solving the set of equations \( \Theta_a = 0 \), \( \Theta_+ = 0 \), and \( \Theta_-=0 \) for the case \( g_a = 0 \). The steady-state values of the variables \( a \), \( p_+ \), and \( p \), are found to be given by \( a_0 = 0 \),
\[ p_+ = \frac{\omega_1}{\gamma_1} \left( \frac{\omega_a^2 - \omega_1^2}{\omega_a^2} \right) \frac{p_0}{1 + \frac{\Lambda_1}{\gamma_1}} + \frac{4\omega_1^2}{\gamma_1}, \tag{A42} \]
\[ p_0 = \frac{\gamma_1}{\omega_a} \left( \frac{\omega_a^2 - \omega_1^2}{\omega_a^2} \right) p_0, \tag{A43} \]
respectively. For the case where \( \gamma_1, \gamma_2 \ll \omega_a \), Eqs. (A37) and (A38) become
\[ \frac{\omega_{\text{fd}}}{\omega_a} = \sqrt{1 - \frac{4\omega_1^2}{\omega_a^2}}, \tag{A44} \]
\[ \frac{\omega_{\text{fd}}}{\omega_a} = \sqrt{1 - \frac{4\omega_1^2}{\omega_a^2}}. \tag{A45} \]

With the help of Eqs. (A36), (A39), (A44), and (A45), one obtains, for this case,
\[ \frac{\Lambda_1}{\omega_a} = \frac{8g_1^2\omega_1}{\omega_a^3} \left( \frac{1}{\gamma_1} + \frac{\gamma_2}{\gamma_1} \right) p_0 \cdot \frac{\omega_a^2 - \omega_1^2}{\omega_a^2 - \frac{\Lambda_1}{\gamma_1} + \frac{4\omega_1^2}{\gamma_1}}, \tag{A46} \]
where
\[ \eta = \frac{2\gamma_1}{\gamma_1} \left( 1 - \frac{\omega_0^2}{\omega_a^2} \right) - i \left( \frac{\omega_0^2}{\omega_a^2} - 1 \right) \tag{A47} \]
and where \( \omega_R = \sqrt{4\omega_1^2 + \Delta_{\text{pe}}^2} \) is the Rabi frequency of the driven spin.

**APPENDIX B: INTERMODE COUPLING**

In general, Eq. (A26) can be employed for calculating the eigenvalue of a cavity mode that is weakly coupled to any given ancilla system. In the previous section, the ancilla system under consideration was an ensemble of driven spins, whereas in the current section the ancilla system is taken to be the driven spiral mode. In general, the second-order term \(-g^2[VR(\omega_1)V]_{11}\) in Eq. (A26) can be calculated by evaluating the steady-state response of the ancilla system to small monochromatic oscillations of the cavity mode at its own resonance frequency. Substituting the steady-state solution into the equation of motion of the cavity mode gives its eigenvalue. This approach will be employed in this section.

The Hamiltonian of the two-mode cavity closed system is taken to be given by
\[ \hbar^{-1}\mathcal{H}_{ab} = \omega_a (A_1^\dagger A_1 + \frac{1}{2}) + \omega_b (A_b^\dagger A_b + \frac{1}{2}) \]
\[ + K (A_a + A_b^\dagger)(A_b + A_a^\dagger)^2, \tag{B1} \]
where \( \omega_a \) and \( A_a \) (\( \omega_b \) and \( A_b \)) are the angular frequency and the annihilation operator, respectively, of the omega (spiral) resonator, and \( K \) is the intermode coupling coefficient. The Heisenberg equations of motion are given by [see Eq. (A24)]
\[ \frac{dA_a}{dt} + i\omega_a A_a + iK(A_b + A_b^\dagger)^2 = 0, \tag{B2} \]
\[ \frac{dA_b}{dt} + i\omega_b A_b + 2iK(A_a + A_a^\dagger)(A_b + A_b^\dagger) = 0. \tag{B3} \]

Adding damping and driving leads to
\[ \frac{dA_a}{dt} + (i\omega_a + \gamma_a)A_a + iK(A_b + A_b^\dagger)^2 = F_a \tag{B4} \]
and
\[ \frac{dA_b}{dt} + (i\omega_b + \gamma_b)A_b + 2iK(A_a + A_a^\dagger)(A_b + A_b^\dagger) = F_b e^{-i(\omega_a + \omega_b)t} + F_b, \tag{B5} \]
where both noise terms \( F_a \) and \( F_b \) have a vanishing expectation value. Averaging yields
\[ \frac{dA_a}{dt} + (i\omega_a + \gamma_a)\langle A_a \rangle + iK\langle A_b + A_b^\dagger \rangle^2 = 0 \tag{B6} \]
and
\[ \frac{dA_b}{dt} + (i\omega_b + \gamma_b)\langle A_b \rangle + S_{b1} + S_{b2} = F_b e^{-i(\omega_a + \omega_b)t}, \tag{B7} \]
where
\[ \langle A_a \rangle = A_a e^{-i\omega_a t}, \tag{B8} \]
\[ \langle A_b \rangle = A_b e^{-i\omega_b t}, \tag{B9} \]
and where
\[ S_{b1} = 2iK(A_a + A_a^\dagger)\langle A_b \rangle, \tag{B10} \]
\[ S_{b2} = 2iK(A_a + A_a^\dagger)\langle A_b^\dagger \rangle. \tag{B11} \]

In the sections below, the effect of the terms \( S_{b1} \) and \( S_{b2} \) is separately evaluated.
1. The effect of the $S_{b1}$ term

When the term $S_{b2}$ is disregarded, Eq. (B7) becomes

$$\frac{dC_b}{dt} + (i\Omega_b + \gamma_b)C_b = F_{bf}. \quad (B12)$$

where

$$\Omega_b = -\omega_D + 2K(A_a + A_a^*), \quad (B13)$$

and where

$$A_b = C_b e^{-i(\omega_s + \alpha t)}t. \quad (B14)$$

By employing the notation

$$C_b = C_{b0} + C_b, \quad (B15)$$

where

$$C_{b0} = \frac{F_{bf}}{-i\omega_D + \gamma_b}, \quad (B16)$$

one obtains, in the limit of small $K$,

$$\frac{dC_b}{dt} + (-i\omega_D + \gamma_b)C_b = -2iK(A_a + A_a^*)C_{b0}. \quad (B17)$$

Let $A_b = a_b e^{-i\omega_b t}$ [see Eq. (B8)] and assume that $a_b$ is constant. The steady-state solution reads

$$c_b = \frac{2iK C_{b0} A_a}{i(\omega_a + \omega_s) - \gamma_b} + \frac{2iK C_{b0} A_a^*}{i(\omega_a - \omega_s) - \gamma_b}. \quad (B18)$$

When only terms proportional to $A_a$ are kept, one finds the coupling term in Eq. (B6) can be expressed

$$iK(A_a + A_a^*)^2 \simeq \frac{4iK^2 |C_{b0}|^2 \omega_D A_a}{[i(\omega_a - \omega_D) - \gamma_b][i(\omega_a + \omega_D) - \gamma_b]} \quad (B19)$$

2. The effect of the $S_{b2}$ term

For this case, the term $S_{b1}$ in Eq. (B7) is disregarded. Furthermore, the counter-rotating term proportional to $A_a^* A_a^*$ is disregarded as well [see Eq. (B11)]. For this case, Eq. (B7) becomes

$$\frac{d\alpha_b}{dt} + \gamma_b \alpha_b + 2iKA_a a_b^* e^{i\omega_b t} = F_{bf} e^{-i\omega_b t}. \quad (B20)$$

where

$$\omega_s = 2\omega_b - \omega_a. \quad (B21)$$

Consider a solution of Eq. (B20) having the form [26]

$$\alpha_b = \alpha e^{-i\omega_b t} + \beta e^{i(\omega_s + \omega_b)t}. \quad (B22)$$

Substituting the solution into Eq. (B20) and assuming that $\alpha$, $\beta$, and $a_b$ are all constants lead to

$$(-i\omega_D + \gamma_b)\alpha + 2iKA_a \beta^* = F_{bf} \quad (B23)$$

and

$$[i(\omega_s + \omega_D) + \gamma_b]\beta + 2iKA_a^* \alpha = 0, \quad (B24)$$

thus

$$\alpha = -\frac{F_{bf}}{-i\omega_D + \gamma_b - \frac{4K^2 |A_a|^2}{i(\omega_s + \omega_D) + \gamma_b}}. \quad (B25)$$

and

$$\beta = \frac{-2iKA_a^*}{i(\omega_s + \omega_D) + \gamma_b}. \quad (B26)$$

The steady-state solution (B22) can be used to express the coupling term $iK(A_b + A_b^*)^2$ in Eq. (B6) in terms of $A_a$. To that end, $A_b$ is expressed as [see Eqs. (B8), (B22), and (B26)]

$$A_b = \alpha e^{-i(\omega_b + \omega_s)t} + \beta e^{i(\omega_s + \omega_D - \omega_b)t}$$

$$= \alpha e^{-i(\omega_s + \omega_D)t} - 2iKA_a^* e^{i(\omega_b + \omega_s)t} - \gamma_b A_a. \quad (B27)$$

When only terms proportional to $A_a$ are kept, the following approximation is employed [see Eq. (B25)]:

$$\alpha \simeq \frac{F_{bf}}{-i\omega_D + \gamma_b}. \quad (B28)$$

and it is assumed that $|\omega_D| \ll |\omega_s|$ for evaluating $\beta$ [see Eq. (B26)], the coupling term in Eq. (B6) becomes

$$iK(A_b + A_b^*)^2 \simeq -\frac{4K^2 |A_a|^2}{i(\omega_s + \omega_D) + \gamma_b} \simeq -\frac{4K^2 |F_{bf}|^2 A_a^2}{(i\omega_s + \gamma_b)(\omega_a^2 + \omega_s^2)} \quad (B29)$$

When $\gamma_b \ll \omega_s$, one has

$$iK(A_b + A_b^*)^2 \simeq \frac{4K^2 |F_{bf}|^2 (i\omega_s - \gamma_b) A_a^2}{\omega_a^2 (\omega_a^2 + \omega_s^2)}. \quad (B30)$$


