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MEMS - Lecture Notes

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Technion

Preface

To be written...

Contents

1. Euler-Lagrange Equations

Purely mechanical systems in the classical limit can be described and analyzed using Newton's theory of Mechanics. Similarly, maxwell's theory allows treating purely electromagnetic classical systems. However, a typical MEMS device combines both mechanical and electromagnetic elements. For such systems it is convenient to employ the Hamilton's formalism of classical physics. This formalism unites the main laws of classical physics into a single framework. This chapter discusses the derivation of Euler-Lagrange equations from the principle of least action. The solution of these equations of motion yields the time evolution of the system under study The Hamilton's formalism also allows a relatively simple description of the laws of classical statistical mechanics, which will be discussed in chapter 3.

1.1 Action and Lagrangian

Consider a classical physical system having N degrees of freedom. The classical state of the system can be described by N independent coordinates q_n, where $n = 1, 2, \cdots, N$. The vector of coordinates is denoted by

$$Q = (q_1, q_2, \cdots, q_N) \ . \tag{1.1}$$

Consider the case where the vector of coordinates takes the value Q_1 at time t_1 and the value Q_2 at a later time $t_2 > t_1$, namely

$$Q(t_1) = Q_1 \ , \tag{1.2}$$

$$Q(t_2) = Q_2 \ . \tag{1.3}$$

The *action* S associated with the evolution of the system from time t_1 to time t_2 is defined by

$$S = \int_{t_1}^{t_2} \mathrm{d}t\, \mathcal{L} \ , \tag{1.4}$$

where $\mathcal{L}$ is the Lagrangian function of the system. In general, the Lagrangian is a function of the coordinates Q, the velocities $\dot{Q}$ and time t, namely

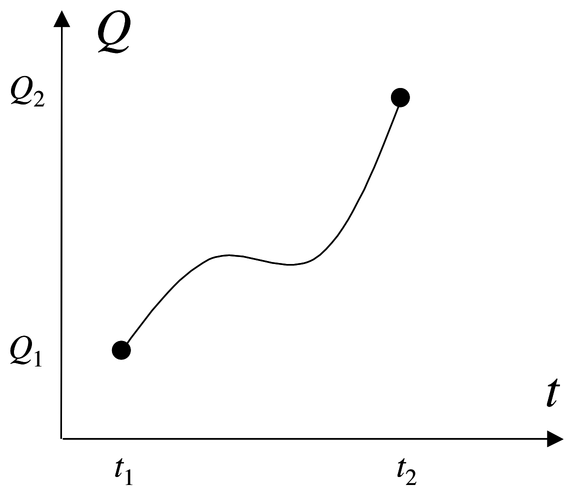


Fig. 1.1. A trajectory taken by the system from point Q_1 at time t_1 to point Q_2 at time t_2.

$$\mathcal{L} = \mathcal{L}\left(Q, \dot{Q}; t\right) , \tag{1.5}$$

where

$$\dot{Q} = (\dot{q}_1, \dot{q}_2, \cdots, \dot{q}_N) , \tag{1.6}$$

and where overdot denotes time derivative. The time evolution of Q, in turn, depends of the trajectory taken by the system from point Q_1 at time t_1 to point Q_2 at time t_2 (see Fig. 1.1). For a given trajectory Γ the time dependency is denoted as

$$Q(t) = Q_\Gamma(t) . \tag{1.7}$$

1.2 Principle of Least Action

For any given trajectory $Q(t)$ the action can be evaluated using Eq. (1.4). Consider a classical system evolving in time from point Q_1 at time t_1 to point Q_2 at time t_2 along the trajectory $Q_\Gamma(t)$. The trajectory $Q_\Gamma(t)$, which is obtained from the laws of classical physics, has the following unique property known as the principle of least action:

Proposition 1.2.1 (principle of least action). *Among all possible trajectories from point Q_1 at time t_1 to point Q_2 at time t_2 the action obtains its minimal value by the classical trajectory $Q_\Gamma(t)$.*

In a weaker version of this principle, the action obtains a local minimum for the trajectory $Q_\Gamma(t)$. As the following theorem shows, the principle of least action leads to a set of equations of motion, known as Euler-Lagrange equations.

Theorem 1.2.1. *The classical trajectory $Q_\Gamma(t)$, for which the action obtains its minimum value, obeys the Euler-Lagrange equations of motion, which are given by*

$$\frac{\mathrm{d}}{\mathrm{d}t}\frac{\partial \mathcal{L}}{\partial \dot{q}_n} = \frac{\partial \mathcal{L}}{\partial q_n} \;, \tag{1.8}$$

where $n = 1, 2, \cdots, N$.

Proof. Consider another trajectory $Q_{\Gamma'}(t)$ from point Q_1 at time t_1 to point Q_2 at time t_2 (see Fig. 1.2). The difference

$$\delta Q = Q_{\Gamma'}(t) - Q_\Gamma(t) = (\delta q_1, \delta q_2, \cdots, \delta q_N) \tag{1.9}$$

is assumed to be infinitesimally small. To lowest order in δQ the change in the action δS is given by

$$\begin{aligned}\delta S &= \int_{t_1}^{t_2} \mathrm{d}t\; \delta \mathcal{L} \\ &= \int_{t_1}^{t_2} \mathrm{d}t\; \left[\sum_{n=1}^{N} \frac{\partial \mathcal{L}}{\partial q_n}\delta q_n + \sum_{n=1}^{N} \frac{\partial \mathcal{L}}{\partial \dot{q}_n}\delta \dot{q}_n\right] \\ &= \int_{t_1}^{t_2} \mathrm{d}t\; \left[\sum_{n=1}^{N} \frac{\partial \mathcal{L}}{\partial q_n}\delta q_n + \sum_{n=1}^{N} \frac{\partial \mathcal{L}}{\partial \dot{q}_n}\frac{\mathrm{d}}{\mathrm{d}t}\delta q_n\right] \;.\end{aligned} \tag{1.10}$$

Integrating the second term by parts leads to

$$\begin{aligned}\delta S = &\int_{t_1}^{t_2} \mathrm{d}t\; \sum_{n=1}^{N} \left(\frac{\partial \mathcal{L}}{\partial q_n} - \frac{\mathrm{d}}{\mathrm{d}t}\frac{\partial \mathcal{L}}{\partial \dot{q}_n}\right)\delta q_n \\ &+ \sum_{n=1}^{N} \left[\frac{\partial \mathcal{L}}{\partial \dot{q}_n}\delta q_n\right|_{t_1}^{t_2} \;.\end{aligned} \tag{1.11}$$

The last term vanishes since

$$\delta Q(t_1) = \delta Q(t_2) = 0 \;. \tag{1.12}$$

The principle of least action implies that

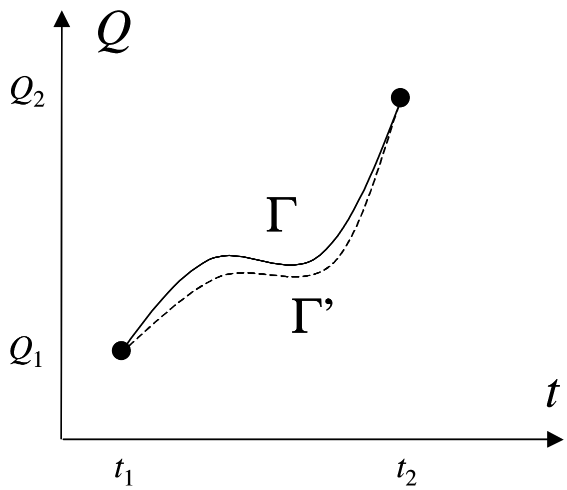


Fig. 1.2. The classical trajectory $Q_{\Gamma}(t)$ and the trajectory $Q_{\Gamma'}(t)$.

$$\delta S = 0 \; . \tag{1.13}$$

This has to be satisfied for *any* δQ, therefore the following must hold

$$\frac{\mathrm{d}}{\mathrm{d}t}\frac{\partial \mathcal{L}}{\partial \dot{q}_n} = \frac{\partial \mathcal{L}}{\partial q_n} \; . \tag{1.14}$$

In what follows we will assume for simplicity that the kinetic energy T of the system can be expressed as a function of the velocities $\dot{Q}$ only (namely, it does not explicitly depend on the coordinates Q). The components of the generalized force F_n, where $n = 1, 2, \cdots, N$, are derived from the potential energy U of the system as follows

$$F_n = -\frac{\partial U}{\partial q_n} + \frac{\mathrm{d}}{\mathrm{d}t}\frac{\partial U}{\partial \dot{q}_n} \; . \tag{1.15}$$

When the potential energy can be expressed as a function of the coordinates Q only (namely, when it is independent on the velocities $\dot{Q}$), the system is said to be *conservative*. For that case, the Lagrangian can be expressed in terms of T and U as

$$\mathcal{L} = T - U \; . \tag{1.16}$$

Example 1.2.1. Consider a point particle having mass m moving in a one-dimensional potential $U(x)$. The Lagrangian is given by

$$\mathcal{L} = T - U = \frac{m\dot{x}^2}{2} - U(x) \; . \tag{1.17}$$

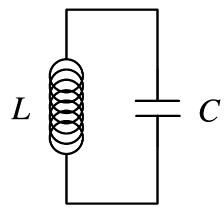


Fig. 1.3. LC resonator.

From the Euler-Lagrange equation

$$\frac{\mathrm{d}}{\mathrm{d}t}\frac{\partial \mathcal{L}}{\partial \dot{x}} = \frac{\partial \mathcal{L}}{\partial x} \,, \tag{1.18}$$

one finds that

$$m\ddot{x} = -\frac{\partial U}{\partial x} \,. \tag{1.19}$$

1.3 Problems

1. Consider an LC resonator made of a capacitor having capacitance C in parallel with an inductor having inductance L (see Fig. 1.3). The state of the system is characterized by the coordinate q , which is the charge stored by the capacitor. Find the Euler-Lagrange equation of the system.
2. Show that the Lagrange equations are coordinate invariant.

1.4 Solutions

1. The kinetic energy in this case $T = L\dot{q}^2/2$ is the energy stored in the inductor, and the potential energy $U = q^2/2C$ is the energy stored in the capacitor. The Lagrangian is given by

$$\mathcal{L} = T - U = \frac{L\dot{q}^2}{2} - \frac{q^2}{2C} \,. \tag{1.20}$$

The Euler-Lagrange equation for the coordinate q is given by

$$\frac{\mathrm{d}}{\mathrm{d}t}\frac{\partial \mathcal{L}}{\partial \dot{q}} = \frac{\partial \mathcal{L}}{\partial q} \,, \tag{1.21}$$

thus

$$L\ddot{q} + \frac{q}{C} = 0 \,. \tag{1.22}$$

This equation expresses the requirement that the voltage across the capacitor is the same as the one across the inductor.

2. Let $\mathcal{L} = \mathcal{L}\left(Q, \dot{Q}; t\right)$ be a Lagrangian of a system, where $Q = (q_1, q_2, \cdots)$ is the vector of coordinates, $\dot{Q} = (\dot{q}_1, \dot{q}_2, \cdots)$ is the vector of velocities, and where overdot denotes time derivative. Consider the coordinates transformation

$$x_a = x_a (q_1, q_2, ..., t) \ , \tag{1.23}$$

where $a = 1, 2, \cdots$. The following holds

$$\dot{x}_a = \frac{\partial x_a}{\partial q_b}\dot{q}_b + \frac{\partial x_a}{\partial t} \ , \tag{1.24}$$

where the summation convention is being used, namely, repeated indices are summed over. Moreover

$$\frac{\partial \mathcal{L}}{\partial q_a} = \frac{\partial \mathcal{L}}{\partial x_b}\frac{\partial x_b}{\partial q_a} + \frac{\partial \mathcal{L}}{\partial \dot{x}_b}\frac{\partial \dot{x}_b}{\partial q_a} \ , \tag{1.25}$$

and

$$\frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{\partial \mathcal{L}}{\partial \dot{q}_a}\right) = \frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{\partial \mathcal{L}}{\partial \dot{x}_b}\frac{\partial \dot{x}_b}{\partial \dot{q}_a}\right) \ . \tag{1.26}$$

As can be seen from Eq. (1.24), one has

$$\frac{\partial \dot{x}_b}{\partial \dot{q}_a} = \frac{\partial x_b}{\partial q_a} \ . \tag{1.27}$$

Thus, using Eqs. (1.25) and (1.26) one finds

$$\begin{aligned}\frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{\partial \mathcal{L}}{\partial \dot{q}_a}\right) - \frac{\partial \mathcal{L}}{\partial q_a} &= \frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{\partial \mathcal{L}}{\partial \dot{x}_b}\frac{\partial x_b}{\partial q_a}\right) \\ &\quad - \frac{\partial \mathcal{L}}{\partial x_b}\frac{\partial x_b}{\partial q_a} - \frac{\partial \mathcal{L}}{\partial \dot{x}_b}\frac{\partial \dot{x}_b}{\partial q_a} \\ &= \left[\frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{\partial \mathcal{L}}{\partial \dot{x}_b}\right) - \frac{\partial \mathcal{L}}{\partial x_b}\right]\frac{\partial x_b}{\partial q_a} \\ &\quad + \left[\frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{\partial x_b}{\partial q_a}\right) - \frac{\partial \dot{x}_b}{\partial q_a}\right]\frac{\partial \mathcal{L}}{\partial \dot{x}_b} \ .\end{aligned} \tag{1.28}$$

As can be seen from Eq. (1.24), the second term vanishes since

$$\frac{\partial \dot{x}_b}{\partial q_a} = \frac{\partial^2 x_b}{\partial q_a \partial q_c}\dot{q}_c + \frac{\partial^2 x_b}{\partial t \partial q_a} = \frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{\partial x_b}{\partial q_a}\right) \ ,$$

thus

$$\frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{\partial \mathcal{L}}{\partial \dot{q}_a}\right) - \frac{\partial \mathcal{L}}{\partial q_a} = \left[\frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{\partial \mathcal{L}}{\partial \dot{x}_b}\right) - \frac{\partial \mathcal{L}}{\partial x_b}\right]\frac{\partial x_b}{\partial q_a} \ . \tag{1.29}$$

The last result shows that if the coordinate transformation is reversible, namely if $\det(\partial x_b / \partial q_a) \neq 0$ then Lagrange equations are coordinate invariant.

2. Capacitive Actuation

This chapter deals with a relatively simple example of an electromechanical system, namely the capacitively actuated point mass. The equations of motion are derived from the Lagrangian of the system and the phenomenon of pull-in is demonstrated.

2.1 Equation of Motion

Consider the electromechanical resonator seen in Fig. 2.1. The mass m can move along the x axis in one dimension. One side of a spring having a spring constant k is attached to the mass, whereas the other side is harnessed to a fixed point on a wall. Let $C(x)$ be the displacement dependent capacitance between the mass and the wall (both mass and wall are assumed to be made of a conducting material, whereas the spring is insulating). Assume that this capacitance can be calculated using the parallel plates capacitance formula $C(x) = \varepsilon_0 A/(d_0 - x)$, where ε_0 is the permittivity constant and A is the effective area. In addition, a voltage source $V(t)$ is connected between the mass and the wall.

Exercise 2.1.1. Find the Lagrangian of the system.

Solution 2.1.1. The state of the system is described using the mechanical displacement coordinate x and the coordinate q, which represents the charge on the capacitor. The kinetic energy of the mechanical element is $m\dot{x}^2/2$, the potential energy of the mechanical element is $kx^2/2$ and the potential energy of the capacitor is $q^2/2C$. It is important to take also into account the potential energy of the voltage source. The voltage source can be treated as a charged capacitor having capacitance C_s. The change in the potential energy of the source U_s due to change in its charge from an initial value of q_0 to the value $q_0 - q$ is given by

$$\delta U_\mathrm{s} = \frac{q_0^2}{2C_\mathrm{s}} - \frac{(q_0 - q)^2}{2C_\mathrm{s}} = \frac{q_0}{C_\mathrm{s}} q \left(1 - \frac{q}{2q_0}\right) . \tag{2.1}$$

While the initial voltage across the source is given by $V = q_0/C_\mathrm{s}$, the voltage when the charge is $q_0 - q$ is $(q_0 - q)/C_\mathrm{s}$. The ability of the voltage source

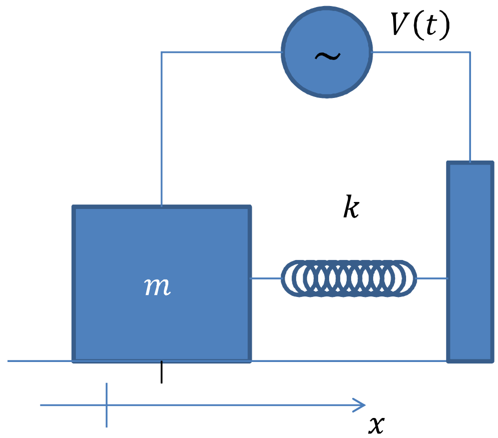


Fig. 2.1. EMR

to supply charge q while keeping the voltage almost unchanged implies that $q \ll q_0$, and consequently the change in the potential energy of the source is approximately given by

$$\delta U_{\mathrm{s}} = Vq \; . \tag{2.2}$$

Thus the Lagrangian of the system is given by

$$\mathcal{L} = T - U = \frac{m\dot{x}^2}{2} - \frac{kx^2}{2} - \frac{q^2}{2C} - qV \; . \tag{2.3}$$

Exercise 2.1.2. Derive the Euler-Lagrange equations for the system.

Solution 2.1.2. Using Eq. (1.8) one finds that

$$m\ddot{x} = -kx - \frac{q^2}{2}\frac{\partial C^{-1}}{\partial x} \; , \tag{2.4}$$

and

$$0 = -\frac{q}{C} - V \; . \tag{2.5}$$

Using $C(x) = \varepsilon_0 A/(d_0 - x)$ and the notation

$$\omega_{\mathrm{m0}} = \sqrt{\frac{k}{m}} \; , \; C_0 = \frac{\varepsilon_0 A}{d_0} \; , \tag{2.6}$$

the equation of motion for x becomes

$$\ddot{x} + \omega_{\mathrm{m0}}^2 x = \frac{\varepsilon_0 A V^2}{2m(d_0 - x)^2} \; . \tag{2.7}$$

2.2 Static Deflection and Pull-in

Consider the case where the voltage V is time independent. In steady state, when x is time independent, Eq. (2.7) implies that

$$m\omega_{\mathrm{m0}}^2 x = \frac{\varepsilon_0 A V^2}{2\left(d_0 - x\right)^2} \ . \tag{2.8}$$

While the left side of the above relation is the restoring force of the spring, the right hand side expresses the attractive force between the mass and the wall due to the capacitive coupling, which can be evaluated by taking the derivative of the capacitive energy $q^2/2C$ with respect to the displacement x

$$-\frac{\mathrm{d}}{\mathrm{d}x}\frac{q^2}{2C\left(x\right)} = \frac{\varepsilon_0 A V^2}{2\left(d_0 - x\right)^2} \ . \tag{2.9}$$

Let $\zeta = x/d_0$ be the normalized displacement, and let

$$V_{\mathrm{PI}} = \sqrt{\frac{8kd_0^3}{27\varepsilon_0 A}} \ . \tag{2.10}$$

Using this notation Eq. (2.8) becomes

$$\zeta\left(1-\zeta\right)^2 = \frac{4}{27}\left(\frac{V}{V_{\mathrm{PI}}}\right)^2 \ . \tag{2.11}$$

As can be seen from Fig. 2.2, the term $\zeta\left(1-\zeta\right)^2$ obtains a local maxima point at $\zeta = 1/3$. At that point $\zeta\left(1-\zeta\right)^2 = 4/27$. Thus, Eq. (2.11) implies that for voltage V larger than the voltage V_{PI}, which is called the pull-in voltage, the system does not have a steady state solution. When $V > V_{\mathrm{PI}}$ the attractive capacitive force overcomes the restoring force of the spring, and consequently a pull-in (or stiction) occurs, i.e. the mass collapses on the wall.

2.3 Problems

1. Consider the case where the applied voltage V is given by

$$V = V_0 + V_1\left(t\right) \ , \tag{2.12}$$

where V_0 is a constant, $V_1\left(t\right)$ is time dependent, and where

$$\left|V_1\right| \ll \left|V_0\right| \ll V_{\mathrm{PI}} \ . \tag{2.13}$$

Find an approximated equation of motion for x.

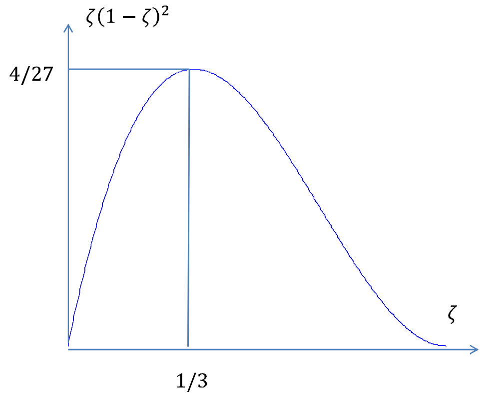


Fig. 2.2. The function $\zeta(1-\zeta)^2$.

2. Consider the actuator that is seen in Fig. 2.3. The mass m can move along the x axis in one dimension. One side of a spring having a spring constant k is attached to the mass, whereas the other side is harnessed to a fixed point on a wall. Let $C(x)$ be the displacement dependent capacitance between the mass and the wall (both mass and wall are assumed to be made of a conducting material, whereas the spring is insulating). Assume that this capacitance can be calculated using the parallel plates capacitance formula $C(x) = \varepsilon_0 A/(d_0 - x)$, where ε_0 is the permittivity constant and A is the effective area. In addition, a voltage source V (which is assumed to be time independent) is connected between the mass and the wall. A fixed capacitor having capacitance C_{s} is serially connected between the voltage source and the 'wall'. Find an expression that relates the static deflection of the mass to the applied voltage V (which is assumed to be time independent). Under what conditions pull-in occurs?

2.4 Solutions

1. The equation of motion (2.7) can be rewritten in a dimensionless form as

$$\frac{\mathrm{d}^2\zeta}{\mathrm{d}\tau^2} + \zeta = \frac{4v^2}{27(1-\zeta)^2} , \tag{2.14}$$

where the dimensionless displacement ζ is given by $\zeta = x/d_0$, the dimensionless time τ is given by $\tau = \omega_{\mathrm{m0}} t$ and the dimensionless voltage v is given by $v = V/V_{\mathrm{PI}}$. The following holds

$$\frac{1}{(1-\zeta)^2} = 1 + 2\zeta + O\left(\zeta^2\right) , \tag{2.15}$$

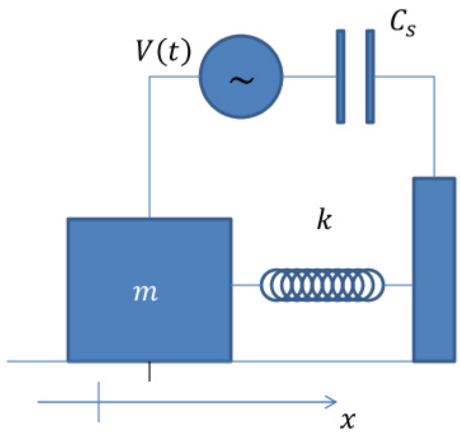


Fig. 2.3.

thus when both ζ and $|V_1/V_0|$ can be assumed to be small one has

$$\frac{4v^2}{27(1-\zeta)^2} \simeq \frac{4v_0^2\left(1+\frac{v_1(t)}{v_0}\right)^2(1+2\zeta)}{27} \simeq \frac{4v_0^2}{27}+\frac{8v_0^2\zeta}{27}+\frac{8v_0^2}{27}\frac{v_1(t)}{v_0}\ , \tag{2.16}$$

where

$$v_0=\frac{V_0}{V_{\mathrm{PI}}}\ ,\ v_1=\frac{V_1}{V_{\mathrm{PI}}}\ ,$$

and thus the equation of motion (2.14) approximately becomes

$$\frac{\mathrm{d}^2(\zeta-\zeta_0)}{\mathrm{d}\tau^2}+(\zeta-\zeta_0)\left(1-\frac{8v_0^2}{27}\right)=\frac{8v_0v_1(t)}{27}\ , \tag{2.17}$$

where

$$\zeta_0=\frac{\frac{4v_0^2}{27}}{1-\frac{8v_0^2}{27}}\simeq\frac{4v_0^2}{27}\ . \tag{2.18}$$

In terms of x and t the last result can be written as

$$\ddot{x}_{\mathrm{d}}+\omega_{\mathrm{m}}^2x_{\mathrm{d}}=f(t)\ , \tag{2.19}$$

where

$$\omega_{\mathrm{m}} = \omega_{\mathrm{m0}} \sqrt{1 - \frac{8}{27} \left(\frac{V_0}{V_{\mathrm{PI}}} \right)^2} , \tag{2.20}$$

$$f(t) = \frac{8\omega_{\mathrm{m0}}^2 d_0 v_0 v_1(t)}{27} , \tag{2.21}$$

$$x_{\mathrm{d}} = x - x_0 , \tag{2.22}$$

and

$$x_0 = \frac{4d_0}{27} \left(\frac{V_0}{V_{\mathrm{PI}}} \right)^2 . \tag{2.23}$$

The length x_0 represents the static displacement towards the wall due to the capacitive coupling, whereas ω_{m} represents, as we will be seen later, the effective angular resonance frequency.

2. The displacement of the mass x and the charge q of both capacitors are taken to be the coordinates of the system. The Lagrangian of the system is given by [see Eq. (2.3)]

$$\mathcal{L} = \frac{m\dot{x}^2}{2} - \frac{kx^2}{2} - \frac{q^2}{2C(x)} - \frac{q^2}{2C_{\mathrm{s}}} - qV . \tag{2.24}$$

The Euler-Lagrange equations (1.8) for the system are given by

$$m\ddot{x} = -kx - \frac{q^2}{2} \frac{\partial C^{-1}}{\partial x} , \tag{2.25}$$

and

$$q \left(\frac{1}{C} + \frac{1}{C_{\mathrm{s}}} \right) = -V , \tag{2.26}$$

thus

$$m\ddot{x} = -kx - \frac{V^2}{2\left(\frac{1}{C} + \frac{1}{C_{\mathrm{s}}} \right)^2} \frac{\partial C^{-1}}{\partial x} . \tag{2.27}$$

In steady state, i.e. when $\dot{x} = 0$, one has in dimensionless form

$$g(\zeta) = \frac{v^2}{2} , \tag{2.28}$$

where

$$\zeta = \frac{x}{d_0} , \; v = \frac{V}{\sqrt{\frac{kd_0^3}{\varepsilon_0 A}}} , \tag{2.29}$$

$$\beta = \frac{C(d_0)}{C_s} = \frac{\varepsilon_0 A}{d_0 C_s} , \tag{2.30}$$

and where

$$g(\zeta) = \zeta(1 - \zeta + \beta)^2 , \tag{2.31}$$

The function $g(\zeta)$ has a local maxima point at $\zeta = (1 + \beta)/3$. For $\beta < 2$ at that point pull-in occurs when $v = \sqrt{2g(\zeta)}$, however, for $\beta \geq 2$ no pull-in occurs since for that case $g(\zeta)$ has no local maxima in the entire accessible range for the mass (i.e. the range $\zeta \leq 1$).

3. The Forced and Damped Harmonic Oscillator

The equation of motion of the capacitively actuated point mass when damping is disregarded has the form [see Eq. (2.19)]

$$\ddot{x} + \omega_{\mathrm{m}}^2 x = f(t) \ , \tag{3.1}$$

where ω_{m} is the effective angular resonance frequency and where $f(t)$ is the forcing term due to externally applied time dependent voltage. Damping can be taken into account by adding a term proportional to the velocity $\dot{x}$.

3.1 Exact Solution

The equation of motion of the forced and damped harmonic oscillator is taken to be given by

$$\ddot{x} + 2\gamma\dot{x} + \omega_0^2 x = f(t) \ , \tag{3.2}$$

where γ is the damping constant. The general solution can be expressed as a sum of a general solution for the homogeneous equation (i.e. the equation for the case $f(t) = 0$) and a particular solution to the nonhomogeneous equation.

3.1.1 The homogeneous equation

For this case $f(t) = 0$ and thus Eq. (3.2) becomes

$$\ddot{x} + 2\gamma\dot{x} + \omega_0^2 x = 0 \ , \tag{3.3}$$

or

$$\left(\mathcal{D}^2 + 2\gamma\mathcal{D} + \omega_0^2\right) x = 0 \ , \tag{3.4}$$

where

$$\mathcal{D} = \frac{\mathrm{d}}{\mathrm{d}t} \ . \tag{3.5}$$

Using the identity

$$\mathcal{D}^2 + 2\gamma\mathcal{D} + \omega_0^2 = (\mathcal{D} - \Gamma)(\mathcal{D} - \Gamma^*) \ , \tag{3.6}$$

where

$$\Gamma = -\gamma + i\sqrt{\omega_0^2 - \gamma^2} \ , \tag{3.7}$$

the homogeneous equation becomes

$$(\mathcal{D} - \Gamma)(\mathcal{D} - \Gamma^*)\, x = 0 \ . \tag{3.8}$$

In this form it is easy to see that both $e^{\Gamma t}$ and $e^{\Gamma^* t}$ are solutions, and thus the general solution to the homogeneous equation is given by

$$x_{\mathrm{h}}(t) = A_1 e^{\Gamma t} + A_2 e^{\Gamma^* t} \ , \tag{3.9}$$

where both A_1 and A_2 are constants, which are determined by initial conditions.

3.1.2 The case $f(t) = F e^{i\omega_{\mathrm{p}} t}$

For this case Eq. (3.2) reads

$$(\mathcal{D} - \Gamma)(\mathcal{D} - \Gamma^*)\, x = F e^{i\omega_{\mathrm{p}} t} \ , \tag{3.10}$$

where F is a constant. Consider a solution having the form

$$x(t) = \mathcal{A} e^{i\omega_{\mathrm{p}} t} \ . \tag{3.11}$$

where $\mathcal{A}$ is a constant. Substituting into Eq. (3.10) yields

$$\mathcal{A} = \frac{F}{(i\omega_{\mathrm{p}} - \Gamma)(i\omega_{\mathrm{p}} - \Gamma^*)} \ , \tag{3.12}$$

or

$$\mathcal{A} = \frac{F}{-\omega_{\mathrm{p}}^2 + 2i\gamma\omega_{\mathrm{p}} + \omega_0^2} \ . \tag{3.13}$$

Adding this particular solution to the homogeneous solution (3.9) yields the general solution

$$x(t) = \frac{F e^{i\omega_{\mathrm{p}} t}}{(i\omega_{\mathrm{p}} - \Gamma)(i\omega_{\mathrm{p}} - \Gamma^*)} + x_{\mathrm{h}}(t) \ . \tag{3.14}$$

3.1.3 The case $f(t) = \int_{-\infty}^{+\infty} \mathrm{d}\omega_{\mathrm{p}}\, F(\omega_{\mathrm{p}})\, e^{i\omega_{\mathrm{p}} t}$

By using Eq. (3.14) and exploiting the linearity of the equation of motion one finds that for this case the solution is given by

$$x(t) = \int_{-\infty}^{+\infty} \mathrm{d}\omega_{\mathrm{p}}\, \frac{F(\omega_{\mathrm{p}})\, e^{i\omega_{\mathrm{p}} t}}{(i\omega_{\mathrm{p}} - \Gamma)(i\omega_{\mathrm{p}} - \Gamma^*)} + x_{\mathrm{h}}(t) \ . \tag{3.15}$$

Physical forces are real, and thus one expect that $F(-\omega_{\mathrm{p}}) = F^*(\omega_{\mathrm{p}})$. This condition guarantees that the integral is real.

3.1.4 The case $f(t) = \delta(t - t_0)$

For this case Eq. (3.2) is given by

$$(\mathcal{D} - \mathit{\Gamma})(\mathcal{D} - \mathit{\Gamma}^*)\,x = \delta(t - t_0)\ . \tag{3.16}$$

Using the identity

$$\delta(t - t_0) = \frac{1}{2\pi}\int_{-\infty}^{+\infty} \mathrm{d}\omega_{\mathrm{p}}\ e^{i\omega_{\mathrm{p}}(t-t_0)}\ , \tag{3.17}$$

one finds from Eq. (3.15) by replacing $F(\omega_{\mathrm{p}})$ by $(1/2\pi)\,e^{-i\omega_{\mathrm{p}}t_0}$ that

$$x(t) = \frac{1}{2\pi}\int_{-\infty}^{+\infty} \mathrm{d}\omega_{\mathrm{p}}\ \frac{e^{i\omega_{\mathrm{p}}(t-t_0)}}{(i\omega_{\mathrm{p}} - \mathit{\Gamma})(i\omega_{\mathrm{p}} - \mathit{\Gamma}^*)} + x_{\mathrm{h}}(t)\ . \tag{3.18}$$

In order to evaluate the integral, the residue theorem is used. The integrand has two poles at $-i\mathit{\Gamma} = i\gamma + \sqrt{\omega_0^2 - \gamma^2}$ and at $-i\mathit{\Gamma}^* = i\gamma - \sqrt{\omega_0^2 - \gamma^2}$. Both poles are located in the upper complex half-plane. For the case $t - t_0 \geq 0$ the factor $e^{i\omega_{\mathrm{p}}(t-t_0)}$ is bounded in the upper complex half-plane. Therefore the integration path for this case can be closed via the upper infinite semicircle in the complex plane. The corresponding residue factors are

$$R_1 = \frac{e^{\mathit{\Gamma}(t-t_0)}}{i(\mathit{\Gamma} - \mathit{\Gamma}^*)}\ , \tag{3.19}$$

$$R_2 = \frac{e^{\mathit{\Gamma}^*(t-t_0)}}{i(\mathit{\Gamma} - \mathit{\Gamma}^*)}\ , \tag{3.20}$$

and thus for $t - t_0 \geq 0$ on finds that

$$\begin{aligned}
&\frac{1}{2\pi}\int_{-\infty}^{+\infty} \mathrm{d}\omega_{\mathrm{p}}\ \frac{e^{i\omega_{\mathrm{p}}(t-t_0)}}{(i\omega_{\mathrm{p}} - \mathit{\Gamma})(i\omega_{\mathrm{p}} - \mathit{\Gamma}^*)} \\
&= \frac{e^{\mathit{\Gamma}(t-t_0)} - e^{\mathit{\Gamma}^*(t-t_0)}}{\mathit{\Gamma} - \mathit{\Gamma}^*} \\
&= \frac{e^{-\gamma(t-t_0)}\sin\left(\sqrt{\omega_0^2 - \gamma^2}\,(t - t_0)\right)}{\sqrt{\omega_0^2 - \gamma^2}}\ .
\end{aligned} \tag{3.21}$$

On the other hand for the case where $t - t_0 < 0$ the factor $e^{i\omega_{\mathrm{p}}(t-t_0)}$ is bounded in the lower complex half-plane, which does not contain poles, and therefore, the integral vanishes for this case. Thus for the general case one has

$$x(t) = g(t, t_0) + x_{\mathrm{h}}(t)\ . \tag{3.22}$$

where $g(t, t_0)$, which is given by

$$g(t,t_0) = u(t-t_0)\,\frac{e^{\Gamma(t-t_0)} - e^{\Gamma^*(t-t_0)}}{\Gamma - \Gamma^*}\ . \tag{3.23}$$

where $u(t)$ is the unit step function

$$u(t) = \begin{cases} 1 & \text{for } t > 0 \\ 1/2 & \text{for } t = 0 \\ 0 & \text{for } t < 0 \end{cases}\ , \tag{3.24}$$

is called the Green function of the system.

3.1.5 General $f(t)$

Expressing $f(t)$ as

$$f(t) = \int_{-\infty}^{+\infty} \mathrm{d}t_0\ \delta(t-t_0)\, f(t_0)\ , \tag{3.25}$$

and using Eq. (3.22) yields

$$x(t) = \int_{-\infty}^{+\infty} \mathrm{d}t_0\ g(t,t_0)\, f(t_0) + x_{\mathrm{h}}(t)\ . \tag{3.26}$$

The factor $u(t-t_0)$ in the expression for $g(t,t_0)$ ensures that the principle of causality is not violated, i.e. the displacement $x(t)$ can be affected by the force $f(t_0)$ at time t_0 only for $t \geq t_0$.

Exercise 3.1.1. Given that the displacement $x(t)$ vanishes for $t < 0$, calculate $x(t)$ for $t > 0$ for a force given by $f(t) = f_0 u(t)$, where f_0 is a constant and where where $u(t)$ is the unit step function [see Eq. (3.24)].

Solution 3.1.1. According to Eq. (3.26) one has for $t > 0$

$$x(t) = f_0 \int_0^{+\infty} \mathrm{d}t_0\ g(t,t_0)\ , \tag{3.27}$$

thus [see Eq. (3.23)]

$$\begin{aligned} x(t) &= f_0 \int_0^{t} \mathrm{d}t_0\ \frac{e^{\Gamma(t-t_0)} - e^{\Gamma^*(t-t_0)}}{\Gamma - \Gamma^*} \\ &= \frac{f_0}{\Gamma\Gamma^*}\left(1 - \frac{\Gamma e^{\Gamma^* t} - \Gamma^* e^{\Gamma t}}{\Gamma - \Gamma^*}\right)\ . \end{aligned} \tag{3.28}$$

With the help of Eq. (3.7) one finds that

$$x(t) = \frac{f_0}{\omega_0^2}\left(1 - \frac{\Gamma e^{-i\sqrt{\omega_0^2-\gamma^2}t} - \Gamma^* e^{i\sqrt{\omega_0^2-\gamma^2}t}}{\Gamma - \Gamma^*} e^{-\gamma t}\right)\ . \tag{3.29}$$

Expressing Γ as

$$\Gamma = \omega_0 e^{i\phi_\Gamma} \ , \tag{3.30}$$

where

$$\operatorname{ctg} \phi_\Gamma = -\frac{\gamma}{\sqrt{\omega_0^2 - \gamma^2}} \ , \tag{3.31}$$

leads to

$$x(t) = \frac{f_0}{\omega_0^2} \left(1 - \frac{\sin\left(\phi_\Gamma - \sqrt{\omega_0^2 - \gamma^2} t\right) e^{-\gamma t}}{\sin \phi_\Gamma} \right) . \tag{3.32}$$

3.2 Rotating Frame

Consider a resonator having damping rate γ and angular resonance frequency ω_0, which is externally driven by a time dependent force $f(t)$. As we have seen previously, the equation of motion (3.2) is given by

$$\ddot{x} + 2\gamma\dot{x} + \omega_0^2 x = f(t) \ . \tag{3.33}$$

Using the identity (3.6), which is given by

$$\mathcal{D}^2 + 2\gamma\mathcal{D} + \omega_0^2 = (\mathcal{D} - \Gamma)(\mathcal{D} - \Gamma^*) \ , \tag{3.34}$$

where $\mathcal{D} = \mathrm{d}/\mathrm{d}t$ and where $\Gamma = -\gamma + i\sqrt{\omega_0^2 - \gamma^2}$ [see Eq. (3.7)], the equation of motion can be rewritten as

$$(\mathcal{D} - \Gamma)(\mathcal{D} - \Gamma^*) x = f(t) \ . \tag{3.35}$$

The complex variable a is defined by

$$a = (\mathcal{D} - \Gamma^*) x = \dot{x} - \Gamma^* x \ . \tag{3.36}$$

While the equation of motion for the real variable x (3.35) is of second order, the complex variable a satisfies the following first order equation of motion

$$(\mathcal{D} - \Gamma) a = f(t) \ . \tag{3.37}$$

Consider the case of monochromatic driving, for which $f(t)$ is taken to be given by $Fe^{i\omega_\mathrm{p} t}$, where F is a complex constant and ω_p is a real constant. For this case it is convenient to introduce the transformation into the rotating frame

$$a = Ae^{i\omega_\mathrm{p} t} \ . \tag{3.38}$$

Substituting into Eq. (3.37) yields

$$\dot{A} + (i\omega_{\mathrm{p}} - \Gamma) A = F \ . \tag{3.39}$$

The quality factor Q of the resonator is defined by $Q = \omega_0 / 2\gamma$. For the case of high quality factor (i.e. the case where $\gamma \ll \omega_0$) one has [see Eq. (3.7)]

$$\Gamma \simeq -\gamma + i\omega_0 \ , \tag{3.40}$$

and consequently the equation of motion for A takes the simplified form

$$\dot{A} + [i (\omega_{\mathrm{p}} - \omega_0) + \gamma] A = F \ . \tag{3.41}$$

In what follows, we will assume that both frequency factors in Eq. (3.41), namely the damping rate γ and the detuning factor $\omega_{\mathrm{p}} - \omega_0$, are both much smaller (in absolute value) than ω_{p}.

Exercise 3.2.1. Consider the case where the factor F (which previously was assumed to be a constant) is allowed to vary in time. Solve Eq. (3.41) for that case.

Solution 3.2.1. Using the notation $\Omega = i (\omega_{\mathrm{p}} - \omega_0) + \gamma$ Eq. (3.41) becomes $\dot{A} + \Omega A = F$. Multiplying by the integration factor $e^{\Omega t}$ yields

$$\frac{\mathrm{d}}{\mathrm{d}t} \left(A e^{\Omega t} \right) = F e^{\Omega t} \ , \tag{3.42}$$

and thus one finds by integration (from initial time t_0) that

$$A(t) = A(t_0) e^{\Omega (t_0 - t)} + \int_{t_0}^{t} \mathrm{d}t' \ F(t') e^{\Omega (t' - t)} \ . \tag{3.43}$$

While employing complex amplitudes is convenient for simplifying the equations, it is important to keep in mind that physically the driving force is obviously real. For the case of monochromatic driving, consider the case where the amplitude $f(t)$ is taken to be given by

$$f(t) = F e^{i\omega_{\mathrm{p}} t} + F^* e^{-i\omega_{\mathrm{p}} t} = 2 |F| \cos (\omega_{\mathrm{p}} t + \phi_{\mathrm{F}}) \ , \tag{3.44}$$

where $F = |F| e^{i\phi_{\mathrm{F}}}$, i.e. the complex conjugate of the term $F e^{i\omega_{\mathrm{p}} t}$ is added in order to ensure that $f(t)$ becomes real. After implementing the transformation into the rotating frame (i.e. the transformation $a = A e^{i\omega_{\mathrm{p}} t}$) the added term will contribute a term oscillating at angular frequency $2\omega_{\mathrm{p}}$. In the rotating frame such a rapidly oscillating term is expected to have a small effect on the dynamics on time scales much longer than ω_{p}^{-1}. In the rotating wave approximation (RWA) this oscillating term is disregarded, and consequently the equation of motion (3.41) remains unchanged.

The displacement x can be expressed in term of a as [see Eq. (3.36)]

$$x = \frac{a - a^*}{\Gamma - \Gamma^*} \ . \tag{3.45}$$

For the case of high quality factor [see Eq. (3.40)] x is approximately given by

$$x = \frac{a - a^*}{2i\omega_0} \ . \tag{3.46}$$

In steady state, i.e. when $\dot{A} = 0$, A is given by [see Eq. (3.41)]

$$A = \frac{F}{i(\omega_{\mathrm{p}} - \omega_0) + \gamma} \ . \tag{3.47}$$

Alternatively, by using the notation $F = |F| e^{i\phi_{\mathrm{F}}}$ and the relation

$$\frac{1}{i(\omega_{\mathrm{p}} - \omega_0) + \gamma} = \frac{e^{-i\phi_{\mathrm{R}}}}{\sqrt{(\omega_{\mathrm{p}} - \omega_0)^2 + \gamma^2}} \ , \tag{3.48}$$

where

$$\phi_{\mathrm{R}} = \tan^{-1} \frac{\omega_{\mathrm{p}} - \omega_0}{\gamma} \ , \tag{3.49}$$

i.e.

$$e^{-i\phi_{\mathrm{R}}} = \frac{-i(\omega_{\mathrm{p}} - \omega_0) + \gamma}{\sqrt{(\omega_{\mathrm{p}} - \omega_0)^2 + \gamma^2}} \ , \tag{3.50}$$

one finds that

$$A = \frac{|F| e^{i(\phi_{\mathrm{F}} - \phi_{\mathrm{R}})}}{\sqrt{(\omega_{\mathrm{p}} - \omega_0)^2 + \gamma^2}} \ . \tag{3.51}$$

The displacement x is thus given by

$$x = \frac{Ae^{i\omega_{\mathrm{p}}t} - A^* e^{-i\omega_{\mathrm{p}}t}}{2i\omega_0} = \frac{|F| \sin(\omega_{\mathrm{p}}t + \phi_{\mathrm{F}} - \phi_{\mathrm{R}})}{\omega_0 \sqrt{(\omega_{\mathrm{p}} - \omega_0)^2 + \gamma^2}} \ . \tag{3.52}$$

At resonance, i.e. when $\omega_{\mathrm{p}} = \omega_0$, this becomes

$$x = \frac{|F| \sin(\omega_{\mathrm{p}}t + \phi_{\mathrm{F}})}{\gamma\omega_0} \ , \tag{3.53}$$

thus for this case x oscillates out of phase with respect to the driving force, which is given by Eq. (3.44).

3.3 Problems

1. Consider a forced harmonic oscillator, whose equation of motion is given by

$$\ddot{x} + 2\gamma\dot{x} + \omega_0^2 x = F e^{-i\omega_{\mathrm{p}} t} \,, \tag{3.54}$$

where F is a constant. Consider a solution having the form

$$x = \mathcal{A}(t)\, e^{-i\omega_{\mathrm{p}} t} \,, \tag{3.55}$$

where $\mathcal{A}(t)$ is an envelope function [while x in the above expression is complex, the actual real displacement can be taken to be given by $\operatorname{Re}\left(\mathcal{A}(t)\, e^{-i\omega_{\mathrm{p}} t}\right)$]. In this exercise the equation of motion (3.41) in the slowly varying envelope approximation is derived using a different approach. To that end, assume that the envelop $\mathcal{A}(t)$ function varies slowly on the time scale of ω_{p}^{-1}. Show that this assumption leads to Eq. (3.41).

3.4 Solutions

1. The following holds

$$\dot{x} = \left(-i\omega_{\mathrm{p}}\mathcal{A} + \frac{\mathrm{d}\mathcal{A}}{\mathrm{d}t}\right) e^{-i\omega_{\mathrm{p}} t} \,, \tag{3.56}$$

and

$$\ddot{x} = \left(-\omega_{\mathrm{p}}^2\mathcal{A} - 2i\omega_{\mathrm{p}}\frac{\mathrm{d}\mathcal{A}}{\mathrm{d}t} + \frac{\mathrm{d}^2\mathcal{A}}{\mathrm{d}t^2}\right) e^{-i\omega_{\mathrm{p}} t} \,. \tag{3.57}$$

Substituting into Eq. (3.54) yields

$$-\omega_{\mathrm{p}}^2\mathcal{A} - 2i\omega_{\mathrm{p}}\frac{\mathrm{d}\mathcal{A}}{\mathrm{d}t} + \frac{\mathrm{d}^2\mathcal{A}}{\mathrm{d}t^2} + 2\gamma\left(-i\omega_{\mathrm{p}}\mathcal{A} + \frac{\mathrm{d}\mathcal{A}}{\mathrm{d}t}\right) + \omega_0^2\mathcal{A} = F \,, \tag{3.58}$$

In the slowly varying envelope approximation, the envelop $A(t)$ is assumed to vary slowly on the time scale of ω_{p}^{-1}, i.e. it is assumed that

$$\left|\frac{\mathrm{d}\mathcal{A}}{\mathrm{d}t}\right| \ll \omega_{\mathrm{p}} \left|\mathcal{A}\right| \,, \tag{3.59}$$

and

$$\left|\frac{\mathrm{d}^2\mathcal{A}}{\mathrm{d}t^2}\right| \ll \omega_{\mathrm{p}} \left|\frac{\mathrm{d}\mathcal{A}}{\mathrm{d}t}\right| \,. \tag{3.60}$$

Dropping these small terms from Eq. (3.58) yields

$$\left(\omega_0^2 - \omega_{\mathrm{p}}^2\right)\mathcal{A} - 2i\omega_{\mathrm{p}}\frac{\mathrm{d}\mathcal{A}}{\mathrm{d}t} - 2i\gamma\omega_{\mathrm{p}}\mathcal{A} = F\ . \tag{3.61}$$

In addition, when $|\omega_0 - \omega_{\mathrm{p}}| \ll \omega_{\mathrm{p}}$ it is convenient to employ the near resonance approximation (3.62), which is given by

$$\omega_0^2 - \omega_{\mathrm{p}}^2 \simeq 2\omega_{\mathrm{p}}\left(\omega_0 - \omega_{\mathrm{p}}\right)\ . \tag{3.62}$$

All these results lead to the following first order evolution equation for A

$$\frac{\mathrm{d}A}{\mathrm{d}t} + \left[i\left(\omega_0 - \omega_{\mathrm{p}}\right) + \gamma\right]\mathcal{A} = \frac{iF}{2\omega_{\mathrm{p}}}\ . \tag{3.63}$$

4. Classical Statistical Mechanics

Mechanical resonators are widely employed as sensors for physical parameters such as acceleration, pressure and mass. The sensitivity of such detectors is limited by the laws of statistical mechanics. In this chapter we discuss the classical limit (i.e. the limit where quantum effects can be disregarded) of statistical mechanics. After defining the Hamiltonian, the density function in thermal equilibrium is introduced and the equipartition theorem is derived. These results will be used in the next chapter to evaluate the limits imposed upon the sensitivity of a resonant detector made of a driven mechanical resonator.

4.1 Hamiltonian

The set of Euler-Lagrange equations contains N second order differential equations. In this section we derive an alternative and equivalent set of equations of motion, known as Hamilton-Jacobi equations, that contains twice the number of equations, namely $2N$, however, of first, instead of second, order.

Definition 4.1.1. *The variable canonically conjugate to q_n is defined by*

$$p_n = \frac{\partial \mathcal{L}}{\partial \dot{q}_n} \; . \tag{4.1}$$

Definition 4.1.2. *The Hamiltonian of a physical system is a function of the vector of coordinates Q, the vector of canonical conjugate variables $P = (p_1, p_2, \cdots, p_N)$ and time, namely*

$$\mathcal{H} = \mathcal{H}(Q, P; t) \; , \tag{4.2}$$

is defined by

$$\mathcal{H} = \sum_{n=1}^{N} p_n \dot{q}_n - \mathcal{L} \; , \tag{4.3}$$

where $\mathcal{L}$ is the Lagrangian.

Theorem 4.1.1. *The classical trajectory satisfies the Hamilton-Jacobi equations of motion, which are given by*

$$\dot{q}_n = \frac{\partial \mathcal{H}}{\partial p_n} \,, \tag{4.4}$$

$$\dot{p}_n = -\frac{\partial \mathcal{H}}{\partial q_n} \,, \tag{4.5}$$

where $n = 1, 2, \cdots, N$.

Proof. The differential of $\mathcal{H}$ is given by

$$\begin{aligned}
\mathrm{d}\mathcal{H} &= \mathrm{d}\sum_{n=1}^{N} p_n \dot{q}_n - \mathrm{d}\mathcal{L} \\
&= \sum_{n=1}^{N} \left(\dot{q}_n \mathrm{d}p_n + p_n \mathrm{d}\dot{q}_n - \underbrace{\frac{\partial \mathcal{L}}{\partial q_n}}_{\frac{\mathrm{d}}{\mathrm{d}t}\frac{\partial \mathcal{L}}{\partial \dot{q}_n}} \mathrm{d}q_n - \underbrace{\frac{\partial \mathcal{L}}{\partial \dot{q}_n}}_{p_n} \mathrm{d}\dot{q}_n \right) - \frac{\partial \mathcal{L}}{\partial t}\mathrm{d}t \\
&= \sum_{n=1}^{N} \left(\dot{q}_n \mathrm{d}p_n - \dot{p}_n \mathrm{d}q_n \right) - \frac{\partial \mathcal{L}}{\partial t}\mathrm{d}t \,.
\end{aligned} \tag{4.6}$$

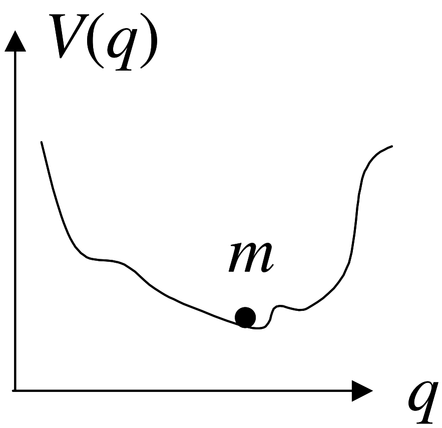
Thus the following holds

$$\dot{q}_n = \frac{\partial \mathcal{H}}{\partial p_n} \,, \tag{4.7}$$

$$\dot{p}_n = -\frac{\partial \mathcal{H}}{\partial q_n} \,, \tag{4.8}$$

$$-\frac{\partial \mathcal{L}}{\partial t} = \frac{\partial \mathcal{H}}{\partial t} \,. \tag{4.9}$$

4.1.1 Example



Consider a particle having mass m in a one dimensional potential $V(q)$. The kinetic energy is given by $T = m\dot{q}^2/2$, thus the canonical conjugate momentum is given by [see Eq. (4.1)] $p = m\dot{q}$. Thus for this example the canonical conjugate momentum equals the mechanical momentum. Note, however, that this is not necessarily always the case. Using the definition (4.3) one finds that the Hamiltonian is given by

$$\begin{aligned} \mathcal{H} &= m\dot{q}^2 - \frac{m\dot{q}^2}{2} + V(q) \\ &= \frac{p^2}{2m} + V(q) \ . \end{aligned} \tag{4.10}$$

Hamilton-Jacobi equations (4.4) and (4.5) read

$$\dot{q} = \frac{p}{m} \tag{4.11}$$

$$\dot{p} = -\frac{\partial V}{\partial q} \ . \tag{4.12}$$

The second equation, which can be rewritten as

$$m\ddot{q} = -\frac{\partial V}{\partial q} \ , \tag{4.13}$$

expresses Newton's second law.

4.2 Density Function

Consider a classical system in thermal equilibrium. The density function $\rho(Q, P)$ is the probability distribution to find the system in the point (Q, P).

The following theorem is given without a proof. Let $\mathcal{H}(Q,P)$ be an Hamiltonian of a system, and assume that $\mathcal{H}$ has the following form

$$\mathcal{H}=\sum_{i=1}^{N}A_i p_i^2+V(Q)\ , \tag{4.14}$$

where A_i are constants. Then in the classical limit, namely in the limit where Plank's constant approaches zero $h\rightarrow 0$, the density function is given by

$$\rho(Q,P)=N\exp\left(-\beta\mathcal{H}(Q,P)\right)\ , \tag{4.15}$$

where

$$N=\frac{1}{\int \mathrm{d}Q\int \mathrm{d}P\ \exp\left(-\beta\mathcal{H}(Q,P)\right)} \tag{4.16}$$

is a normalization constant, $\beta=1/k_{\mathrm{B}}T$, where k_{B} is the Boltzmann's constant and T is the temperature. The notation $\int \mathrm{d}Q$ indicates integration over all coordinates, namely $\int \mathrm{d}Q=\int \mathrm{d}q_1\int \mathrm{d}q_1\cdot ...\cdot\int \mathrm{d}q_N$, and similarly $\int \mathrm{d}P=\int \mathrm{d}p_1\int \mathrm{d}p_1\cdot ...\cdot\int \mathrm{d}p_N$.

Let $A(Q,P)$ be a variable which depends on the coordinates Q and their canonical conjugate momentum variables P. Using the above theorem the average value of A can be calculates as:

$$\begin{aligned}\langle A(Q,P)\rangle&=\int \mathrm{d}Q\int \mathrm{d}P\ A(Q,P)\,\rho(Q,P)\\&=\frac{\int \mathrm{d}Q\int \mathrm{d}P\ A(Q,P)\exp\left(-\beta\mathcal{H}(Q,P)\right)}{\int \mathrm{d}Q\int \mathrm{d}P\ \exp\left(-\beta\mathcal{H}(Q,P)\right)}\ .\end{aligned} \tag{4.17}$$

4.2.1 Equipartition Theorem

Assume that the Hamiltonian has the following form

$$\mathcal{H}=B_i q_i^2+\tilde{\mathcal{H}}\ , \tag{4.18}$$

where B_i is a constant and where $\tilde{\mathcal{H}}$ is independent of q_i. Then the following holds

$$\left\langle B_i q_i^2\right\rangle=\frac{k_{\mathrm{B}}T}{2}\ . \tag{4.19}$$

Similarly, assume that the Hamiltonian has the following form

$$\mathcal{H}=A_i p_i^2+\tilde{\mathcal{H}}\ , \tag{4.20}$$

where A_i is a constant and where $\tilde{\mathcal{H}}$ is independent of p_i. Then the following holds

$$\left\langle A_i p_i^2 \right\rangle = \frac{k_{\mathrm{B}} T}{2} \; . \tag{4.21}$$

To prove the theorem for the first case we use Eq. (4.17)

$$\begin{aligned}
\left\langle B_i q_i^2 \right\rangle &= \frac{\int \mathrm{d}Q \int \mathrm{d}P \; B_i q_i^2 \exp\left(-\beta \mathcal{H}\left(Q, P\right)\right)}{\int \mathrm{d}Q \int \mathrm{d}P \; \exp\left(-\beta \mathcal{H}\left(Q, P\right)\right)} \\
&= \frac{\int \mathrm{d}q_i \; B_i q_i^2 \exp\left(-\beta B_i q_i^2\right)}{\int \mathrm{d}q_i \; \exp\left(-\beta B_i q_i^2\right)} \\
&= -\frac{\partial}{\partial \beta} \log\left(\int \mathrm{d}q_i \; \exp\left(-\beta B_i q_i^2\right)\right) \\
&= -\frac{\partial}{\partial \beta} \log\left(\sqrt{\frac{\pi}{\beta B_i}}\right) \\
&= \frac{1}{2\beta} \; .
\end{aligned} \tag{4.22}$$

The proof for the second case is similar.

Example 4.2.1. Consider a harmonic oscillator made of a particle having mass m in a one dimensional parabolic potential given by $V\left(q\right) = \left(1/2\right) k q^2$, where k is the spring constant. Calculate the average energy of the system.

Solution 4.2.1. The kinetic energy is given by $p^2/2m$, where p is the canonical momentum variable conjugate to q. The Hamiltonian is given by

$$\mathcal{H} = \frac{p^2}{2m} + \frac{kq^2}{2} \; . \tag{4.23}$$

In the classical limit the average energy of the system can be easily calculated using the equipartition theorem

$$U = \left\langle \mathcal{H} \right\rangle = k_{\mathrm{B}} T \; . \tag{4.24}$$

4.3 Problems

1. Show that
$$\frac{\mathrm{d}\mathcal{H}}{\mathrm{d}t} = \frac{\partial \mathcal{H}}{\partial t} \; . \tag{4.25}$$
2. Assume that the kinetic energy of a conservative system is given by
$$T = \sum_{n,m} \alpha_{nm} \dot{q}_n \dot{q}_m \; , \tag{4.26}$$

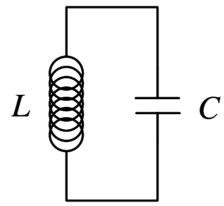


Fig. 4.1. LC resonator

where α_{nm} are constants. Show that the Hamiltonian of the system is given by

$$\mathcal{H} = T + U \,, \tag{4.27}$$

where T is the kinetic energy of the system and where U is the potential energy.

3. Consider a capacitor having capacitance C connected in parallel to an inductor having inductance L (see Fig. 4.1). Let q be the charge stored in the capacitor. Find Hamilton-Jacobi equations for the system.

4.4 Solutions

1. By using Eqs. (4.4) and (4.5) one finds that

$$\frac{\mathrm{d}\mathcal{H}}{\mathrm{d}t} = \sum_{n=1}^{N} \underbrace{\left(\frac{\partial \mathcal{H}}{\partial q_n} \dot{q}_n + \frac{\partial \mathcal{H}}{\partial p_n} \dot{p}_n \right)}_{=0} + \frac{\partial \mathcal{H}}{\partial t} = \frac{\partial \mathcal{H}}{\partial t} \,. \tag{4.28}$$

This result implies that $\mathcal{H}$ is time independent provided that $\mathcal{H}$ does not depend on time explicitly, namely, provided that $\partial \mathcal{H} / \partial t = 0$. This property is referred to as the law of energy conservation. The following exercise below further emphasizes the relation between the Hamiltonian and the total energy of the system.

2. For a conservative system the potential energy is independent on velocities, thus

$$p_l = \frac{\partial \mathcal{L}}{\partial \dot{q}_l} = \frac{\partial T}{\partial \dot{q}_l} \,, \tag{4.29}$$

where $\mathcal{L} = T - U$ is the Lagrangian. The Hamiltonian is thus given by

$$\begin{aligned}
\mathcal{H} &= \sum_{l=1}^{N} p_l \dot{q}_l - \mathcal{L} \\
&= \sum_{l} \frac{\partial T}{\partial \dot{q}_l} \dot{q}_l - (T - U) \\
&= \sum_{l,n,m} \alpha_{nm} \left(\dot{q}_m \underbrace{\frac{\partial \dot{q}_n}{\partial \dot{q}_l}}_{\delta_{nl}} + \dot{q}_n \underbrace{\frac{\partial \dot{q}_m}{\partial \dot{q}_l}}_{\delta_{ml}} \right) \dot{q}_l - T + U \\
&= \underbrace{2 \sum_{n,m} \alpha_{nm} \dot{q}_n \dot{q}_m}_{T} - T + U \\
&= T + U \ .
\end{aligned} \tag{4.30}$$

3. The kinetic energy in this case $T = L\dot{q}^2/2$ is the energy stored in the inductor, and the potential energy $V = q^2/2C$ is the energy stored in the capacitor. The canonical conjugate momentum is given by [see Eq. (4.1)] $p = L\dot{q}$, and the Hamiltonian (4.3) is given by

$$\mathcal{H} = \frac{p^2}{2L} + \frac{q^2}{2C} \ . \tag{4.31}$$

Hamilton-Jacobi equations (4.4) and (4.5) read

$$\dot{q} = \frac{p}{L} \tag{4.32}$$

$$\dot{p} = -\frac{q}{C} \ . \tag{4.33}$$

The second equation, which can be rewritten as

$$L\ddot{q} + \frac{q}{C} = 0 \ , \tag{4.34}$$

expresses the requirement that the voltage across the capacitor is the same as the one across the inductor [see Eq. (1.22)].

5. Resonant Detection

Resonant detection is a widely employed technique in a variety of applications. Consider a mechanical resonator, which is characterized by a resonance frequency ω_0 and damping rate γ. Resonant detection is achieved by coupling the measured physical parameter of interest, which is denoted as $\mathcal{P}$, to the resonator in such a way that ω_0 becomes effectively $\mathcal{P}$ dependent, i.e. $\omega_0 = \omega_0(\mathcal{P})$. In such a configuration $\mathcal{P}$ can be measured by externally driving the resonator, and monitoring its response as a function of time by measuring some output signal $X(t)$. Such a scheme allows a sensitive measurement of the parameter $\mathcal{P}$, provided that the average value of $X(t)$, which is denoted as X_0, strongly depends on ω_0, and provided that ω_0, in turn, strongly depends on $\mathcal{P}$.

5.1 Stationary Random Signals

Consider a complex signal $z(t)$ randomly varying in time. As will be discussed below, the random signal $z(t)$ can be characterized by a variety of statistical properties. In this section it will be assumed that $z(t)$ is stationary. This assumption implies that all statistical properties of $z(t)$ remain unchanged when $z(t)$ is replaced by $z(t - t_0)$, where t_0 is a constant (i.e. when the signal is shifted in time).

5.1.1 Power Spectrum

Let $z_\tau(t)$ be a sampling of the signal $z(t)$ in the time interval $(-\tau/2, \tau/2)$, namely

$$z_\tau(t) = \begin{cases} z(t) & -\tau/2 < t < \tau/2 \\ 0 & \text{else} \end{cases} \quad . \tag{5.1}$$

The signal $z_\tau(t)$ can be expressed in terms of its Fourier transform (FT) $z_\tau(\omega)$ as

$$z_\tau(t) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} \mathrm{d}\omega \; z_\tau(\omega) \, e^{-i\omega t} \; . \tag{5.2}$$

Definition 5.1.1. *The power spectrum* $S_z(\omega)$ *of* $z_\tau(t)$ *is defined by*

$$S_z(\omega) \equiv \lim_{\tau\to\infty} \frac{1}{\tau} \left|z_\tau(\omega)\right|^2 \ . \tag{5.3}$$

Let $O(z(t))$ be a functional of the random signal $z(t)$. The expectation value of $O(z(t))$ is defined by

$$\langle O(z(t))\rangle \equiv \lim_{\tau\to\infty} \frac{1}{\tau} \int_{-\infty}^{+\infty} \mathrm{d}t \ O(z_\tau(t)) \ . \tag{5.4}$$

Claim. The following holds

$$\left\langle |z|^2 \right\rangle = \int_{-\infty}^{\infty} \mathrm{d}\omega \ S_z(\omega) \ . \tag{5.5}$$

Proof. According to the definition (5.4) one has

$$\left\langle |z|^2 \right\rangle = \lim_{\tau\to\infty} \frac{1}{\tau} \int_{-\infty}^{+\infty} \mathrm{d}t \ z_\tau^*(t)\, z_\tau(t) \ , \tag{5.6}$$

thus with the help of Eq. (5.2) one finds that

$$\begin{aligned} \left\langle |z|^2 \right\rangle &= \lim_{\tau\to\infty} \frac{1}{2\pi\tau} \int_{-\infty}^{+\infty} \mathrm{d}t \int_{-\infty}^{\infty} \mathrm{d}\omega' \ z_\tau^*(\omega')\, e^{i\omega' t} \int_{-\infty}^{\infty} \mathrm{d}\omega \ z_\tau(\omega)\, e^{-i\omega t} \\ &= \lim_{\tau\to\infty} \frac{1}{2\pi\tau} \int_{-\infty}^{\infty} \mathrm{d}\omega' \ z_\tau^*(\omega') \int_{-\infty}^{\infty} \mathrm{d}\omega \ z_\tau(\omega) \underbrace{\int_{-\infty}^{+\infty} \mathrm{d}t \ e^{-i(\omega-\omega')t}}_{2\pi\delta(\omega-\omega')} \\ &= \lim_{\tau\to\infty} \frac{1}{\tau} \int_{-\infty}^{\infty} \mathrm{d}\omega \ \left|z_\tau(\omega)\right|^2 \ , \end{aligned} \tag{5.7}$$

thus [see Eq. (5.3)]

$$\left\langle |z|^2 \right\rangle = \int_{-\infty}^{\infty} \mathrm{d}\omega \ S_z(\omega) \ . \tag{5.8}$$

5.1.2 Autocorrelation Function

Definition 5.1.2. *The autocorrelation function* $C_z(t)$ *is defined by*

$$C_z(t') = \langle z^*(t+t')\, z(t)\rangle \ . \tag{5.9}$$

Note that the assumption that $z(t)$ is stationary implies that the quantity $\langle z^*(t+t')\, z(t)\rangle$ is independent on t.

Theorem 5.1.1. *(Wiener-Khinchine Theorem) The following holds*

$$C_z(t') = \int_{-\infty}^{\infty} \mathrm{d}\omega \; e^{i\omega t'} S_z(\omega) \; . \tag{5.10}$$

Proof. According to the definition (5.4) one has

$$C_z(t') = \lim_{\tau\to\infty} \frac{1}{\tau} \int_{-\infty}^{+\infty} \mathrm{d}t \; z_\tau^*(t+t') \, z_\tau(t) \;, \tag{5.11}$$

thus with the help of Eq. (5.2) one finds that

$$\begin{aligned} C_z(t') &= \lim_{\tau\to\infty} \frac{1}{2\pi\tau} \int_{-\infty}^{+\infty} \mathrm{d}t \int_{-\infty}^{\infty} \mathrm{d}\omega' \; z_\tau^*(\omega') \, e^{i\omega'(t+t')} \int_{-\infty}^{\infty} \mathrm{d}\omega \; z_\tau(\omega) \, e^{-i\omega t} \\ &= \lim_{\tau\to\infty} \frac{1}{\tau} \int_{-\infty}^{\infty} \mathrm{d}\omega' \; z_\tau^*(\omega') \, e^{i\omega' t'} \int_{-\infty}^{\infty} \mathrm{d}\omega \; z_\tau(\omega) \underbrace{\frac{1}{2\pi} \int_{-\infty}^{+\infty} \mathrm{d}t \; e^{-i(\omega-\omega')t}}_{\delta(\omega-\omega')} \\ &= \int_{-\infty}^{\infty} \mathrm{d}\omega \; e^{i\omega t'} \lim_{\tau\to\infty} \frac{1}{\tau} \left| z_\tau(\omega) \right|^2 \;, \end{aligned} \tag{5.12}$$

thus [see Eq. (5.3)]

$$C_z(t') = \int_{-\infty}^{\infty} \mathrm{d}\omega \; e^{i\omega t'} S_z(\omega) \; . \tag{5.13}$$

Claim. The following holds

$$\langle z^*(\omega') \, z(\omega) \rangle = 2\pi S_z(\omega) \, \delta(\omega - \omega') \; . \tag{5.14}$$

Proof. Inverting the FT in Eq. (5.2) yields

$$z_\tau(\omega) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} \mathrm{d}t \; z_\tau(t) \, e^{i\omega t} \;, \tag{5.15}$$

thus [see Eq. (5.9)]

$$\begin{aligned} \langle z^*(\omega') \, z(\omega) \rangle &= \frac{1}{2\pi} \int_{-\infty}^{\infty} \mathrm{d}t' \int_{-\infty}^{\infty} \mathrm{d}t \; e^{i(\omega t - \omega' t')} \langle z^*(t') \, z(t) \rangle \\ &= \frac{1}{2\pi} \int_{-\infty}^{\infty} \mathrm{d}t' \int_{-\infty}^{\infty} \mathrm{d}t \; e^{i(\omega t - \omega' t')} C_z(t' - t) \; . \end{aligned} \tag{5.16}$$

The variable transformation $t'' = t' - t$ leads to

$$\begin{aligned}\langle z^*(\omega') z(\omega)\rangle &= \frac{1}{2\pi}\int_{-\infty}^{\infty} \mathrm{d}t' \int_{-\infty}^{\infty} \mathrm{d}t''\ e^{-i\omega' t'} e^{i\omega\left(t'-t''\right)} C_z(t'') \\ &= \int_{-\infty}^{\infty} \mathrm{d}t''\ e^{-i\omega t''} C_z(t'') \underbrace{\frac{1}{2\pi}\int_{-\infty}^{\infty} \mathrm{d}t'\ e^{i\left(\omega-\omega'\right)t'}}_{\delta(\omega-\omega')} ,\end{aligned} \tag{5.17}$$

thus, with the help of Eq. (5.10) one finds that

$$\begin{aligned}\langle z^*(\omega') z(\omega)\rangle &= \int_{-\infty}^{\infty} \mathrm{d}t''\ e^{-i\omega t''} C_z(t'')\, \delta(\omega-\omega') \\ &= \int_{-\infty}^{\infty} \mathrm{d}\omega''\ S_z(\omega'')\, \delta(\omega-\omega') \underbrace{\int_{-\infty}^{\infty} \mathrm{d}t''\ e^{i\left(\omega''-\omega\right)t''}}_{2\pi\delta(\omega''-\omega)} \\ &= 2\pi S_z(\omega)\, \delta(\omega-\omega')\ .\end{aligned} \tag{5.18}$$

5.1.3 Estimator

Let $X(t)$ be a *real* stationary random signal, which is assumed to be given by

$$X(t) = X_0 + X_{\mathrm{N}}(t)\ , \tag{5.19}$$

where X_0 is a real constant and where $X_{\mathrm{N}}(t)$ is a real stationary random signal, which is assumed to have a vanishing expectation value, i.e. $\langle X_{\mathrm{N}}(t)\rangle = 0$. Let $X_\tau(t)$ be a sampling of the signal $X(t)$ in the time interval $(-\tau/2, \tau/2)$, namely

$$X_\tau(t) = \begin{cases} X(t) & -\tau/2 < t < \tau/2 \\ 0 & \text{else} \end{cases}\ . \tag{5.20}$$

Let $\hat{X}_0$ be an estimator of the parameter X_0 (i.e. estimator of the average value of $X(t)$), which is taken to be given by

$$\hat{X}_0 = \frac{1}{\tau}\int_{-\infty}^{\infty} \mathrm{d}t\ X_\tau(t)\ . \tag{5.21}$$

Clearly, $\left\langle \hat{X}_0 \right\rangle = X_0$ (since $\langle X_{\mathrm{N}}(t)\rangle = 0$), and therefore the estimator $\hat{X}_0$ is unbiased, i.e. on average it yields the desired result. However, due to the fluctuating noise the variance $\left(\hat{X}_0 - \left\langle \hat{X}_0 \right\rangle\right)^2$ of the estimator X_0 may have a finite value when the sampling time τ is finite.

Claim. The following holds

$$\lim_{\tau\to\infty} \tau \left(\hat{X}_0 - \left\langle \hat{X}_0 \right\rangle\right)^2 = 2\pi S_{X_\mathrm{N}}(0) \ , \tag{5.22}$$

where $S_{X_\mathrm{N}}(0)$ is the zero frequency power spectrum of $X_\mathrm{N}(t)$.

Proof. Using Eq. (5.19) and the relation $\left\langle \hat{X}_0 \right\rangle = X_0$ one finds that

$$\hat{X}_0 - \left\langle \hat{X}_0 \right\rangle = \frac{1}{\tau} \int_{-\infty}^{\infty} \mathrm{d}t \ X_{\mathrm{N}\tau}(t) \ , \tag{5.23}$$

where $X_{\mathrm{N}\tau}(t)$ is a sampling of $X_\mathrm{N}(t)$ in the time interval $(-\tau/2, \tau/2)$, i.e.

$$X_{\mathrm{N}\tau}(t) = \begin{cases} X_\mathrm{N}(t) & -\tau/2 < t < \tau/2 \\ 0 & \text{else} \end{cases} \ , \tag{5.24}$$

thus, in terms of the autocorrelation function $C_{X_\mathrm{N}}(t)$ of $X_\mathrm{N}(t)$ one has

$$\begin{aligned} \lim_{\tau\to\infty} \tau \left(\hat{X}_0 - \left\langle \hat{X}_0 \right\rangle\right)^2 &= \lim_{\tau\to\infty} \frac{1}{\tau} \left(\int_{-\infty}^{\infty} \mathrm{d}t \ X_{\mathrm{N}\tau}(t) \right)^2 \\ &= \lim_{\tau\to\infty} \frac{1}{\tau} \int_{-\infty}^{\infty} \mathrm{d}t \int_{-\infty}^{\infty} \mathrm{d}t' \ X_{\mathrm{N}\tau}(t) X_{\mathrm{N}\tau}(t') \\ &= \int_{-\infty}^{\infty} \mathrm{d}t'' \lim_{\tau\to\infty} \frac{1}{\tau} \int_{-\infty}^{\infty} \mathrm{d}t' \ X_{\mathrm{N}\tau}(t' + t'') X_{\mathrm{N}\tau}(t') \\ &= \int_{-\infty}^{\infty} \mathrm{d}t'' C_{X_\mathrm{N}}(t'') \ . \end{aligned} \tag{5.25}$$

Finally, the Wiener-Khinchine theorem (5.10) leads to

$$\begin{aligned} \lim_{\tau\to\infty} \tau \left(\hat{X}_0 - \left\langle \hat{X}_0 \right\rangle\right)^2 &= \int_{-\infty}^{\infty} \mathrm{d}t'' \int_{-\infty}^{\infty} \mathrm{d}\omega \ e^{i\omega t''} S_{X_\mathrm{N}}(\omega) \\ &= \int_{-\infty}^{\infty} \mathrm{d}\omega \ S_{X_\mathrm{N}}(\omega) \underbrace{\int_{-\infty}^{\infty} \mathrm{d}t'' \ e^{i\omega t''}}_{=2\pi\delta(\omega)} \ , \end{aligned} \tag{5.26}$$

thus

$$\lim_{\tau\to\infty} \tau \left(\hat{X}_0 - \left\langle \hat{X}_0 \right\rangle\right)^2 = 2\pi S_{X_\mathrm{N}}(0) \ . \tag{5.27}$$

5.2 Mechanical Resonator Coupled to Thermal Bath

Consider a mechanical resonator with mass m, resonance frequency ω_0, and damping rate γ. The equation of motion is given by [see Eq. (3.2)]

$$\ddot{x} + 2\gamma\dot{x} + \omega_0^2 x = f(t) \ , \tag{5.28}$$

where $f(t)$ is the applied force. As can be see from Eq. (3.9), which is given by

$$x_{\mathrm{h}}(t) = A_1 e^{\Gamma t} + A_2 e^{\Gamma^* t} \ , \tag{5.29}$$

where $\Gamma = -\gamma + i\sqrt{\omega_0^2 - \gamma^2}$ and where both A_1 and A_2 are constants, the solution of the homogeneous equation (i.e. the equation for the case $f(t) = 0$) satisfies the following

$$\lim_{t\to\infty} x(t) = 0 \ , \tag{5.30}$$

i.e. in steady state (i.e. in the long time limit) $x(t) = 0$.

On the other hand, when the system is in thermal equilibrium at finite temperature T, one expects according to the equipartition theorem that both $\langle x^2 \rangle$ and $\langle \dot{x}^2 \rangle$ remain finite. The contradiction between Eq. (5.30) and the equipartition theorem can be resolved by assuming that the coupling between the resonator and its environment, which is assumed to be in thermal equilibrium at temperature T, gives rise to a fluctuating force acting on the system. We label this fluctuating force as $f_{\mathrm{N}}(t) e^{i\omega_0 t}$ (as will be seen soon, the factor $e^{i\omega_0 t}$ is added to facilitate the transformation into the rotating frame). In this approach even in the absent of any externally applied force the equation of motion is assumed to be given by

$$\ddot{x} + 2\gamma\dot{x} + \omega_0^2 x = f_{\mathrm{N}}(t) e^{i\omega_0 t} \ . \tag{5.31}$$

The force $f_{\mathrm{N}}(t) e^{i\omega_0 t}$ is assumed to have a vanishing expectation value. The other statistical properties of $f_{\mathrm{N}}(t)$ will be determined below in a way that is consistent with the laws of statistical mechanics. This fluctuating force introduces unavoidable noise that introduces a bound upon the sensitivity of sensors based on mechanical resonators.

5.2.1 Power Spectrum

It is convenient to employ the transformation into the rotating frame. For the present case where no external force is applied the angular frequency of the rotating frame is taken to be ω_0. Let $a = \dot{x} - \Gamma^* x$ [see Eq. (3.36)], and let $a = A e^{i\omega_0 t}$ [see Eq. (3.38)]. For the case of high quality factor (i.e. the case where $\gamma \ll \omega_0$) one has $\Gamma \simeq -\gamma + i\omega_0$ [see Eq. (3.40)]. With this notation Eq. (5.31) becomes [see Eq. (3.41)]

$$\dot{A} + \gamma A = f_{\mathrm{N}}(t) \ . \tag{5.32}$$

The following holds

$$|A|^2 = |a|^2 = |\dot{x} - \Gamma^* x|^2 \simeq |\dot{x} + i\omega_0 x|^2 = \dot{x}^2 + \omega_0^2 x^2 \ , \tag{5.33}$$

thus $(m/2)|A|^2$ is the total energy of the resonator. It is therefore expected according to the equipartition theorem that in thermal equilibrium at temperature T the following should hold [see Eq. (4.24)]

$$\frac{m}{2}\left\langle |A|^2 \right\rangle = k_{\mathrm{B}} T \ . \tag{5.34}$$

Exercise 5.2.1. Show that the power spectrum $S_A(\omega)$ of A is given by

$$S_A(\omega) = \frac{2\gamma k_{\mathrm{B}} T}{\pi m} \frac{1}{\gamma^2 + \omega^2} \ . \tag{5.35}$$

Solution 5.2.1. Let $A_\tau(t)$ be a sampling of the displacement function $A(t)$ in the time interval $(-\tau/2, \tau/2)$ [see Eq. (5.1)], namely

$$A_\tau(t) = \begin{cases} A(t) & -\tau/2 < t < \tau/2 \\ 0 & \text{else} \end{cases} \ . \tag{5.36}$$

The sampling $f_{\mathrm{N}\tau}(t)$ is defined in a similar way. The FT $A_\tau(\omega)$ of $A_\tau(t)$ is defined by [see Eq. (5.2)]

$$A_\tau(t) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} \mathrm{d}\omega \ A_\tau(\omega) e^{-i\omega t} \ , \tag{5.37}$$

and similarly for f_{N}

$$f_{\mathrm{N}\tau}(t) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} \mathrm{d}\omega \ f_{\mathrm{N}\tau}(\omega) e^{-i\omega t} \ . \tag{5.38}$$

Substituting into Eq. (5.32) yields

$$A_\tau(\omega) = R(\omega) f_{\mathrm{N}\tau}(\omega) \ , \tag{5.39}$$

where the frequency response function $R(\omega)$ is given by

$$R(\omega) = \frac{1}{-i\omega + \gamma} \ . \tag{5.40}$$

Taking the absolute value squared leads to a relation between the power spectrum $S_A(\omega)$ of A and the power spectrum of the fluctuating force f_{N} (which is labeled by $S_{f_{\mathrm{N}}}(\omega)$)

$$S_A(\omega) = |R(\omega)|^2 S_{f_{\mathrm{N}}}(\omega) \ . \tag{5.41}$$

In terms of $S_{f_{\mathrm{N}}}(\omega)$ Eq. (5.34) can be rewritten as [see also Eq. (5.5)]

$$\begin{aligned}\frac{2k_{\mathrm{B}}T}{m} &= \left\langle |A|^2 \right\rangle \\ &= \int_{-\infty}^{\infty} \mathrm{d}\omega\ S_A(\omega) \\ &= \int_{-\infty}^{\infty} \mathrm{d}\omega\ |R(\omega)|^2 S_{f_{\mathrm{N}}}(\omega)\ .\end{aligned} \tag{5.42}$$

The function $|R(\omega)|^2 = \left(\omega^2+\gamma^2\right)^{-1}$ has a peak at $\omega = 0$ of width γ. The assumption that near $\omega = 0$, where $|R(\omega)|^2$ peaks, the power spectrum of the fluctuating force $S_{f_{\mathrm{N}}}(\omega)$ is a smooth function of ω on the scale γ leads to the following approximation

$$\begin{aligned}\frac{2k_{\mathrm{B}}T}{m} &\simeq S_{f_{\mathrm{N}}}(0) \int_{-\infty}^{\infty} \mathrm{d}\omega\ |R(\omega)|^2 \\ &= S_{f_{\mathrm{N}}}(0) \underbrace{\int_{-\infty}^{\infty} \frac{\mathrm{d}\omega}{\gamma^2+\omega^2}}_{\pi/\gamma} ,\end{aligned} \tag{5.43}$$

thus

$$S_{f_{\mathrm{N}}}(0) = \frac{2\gamma k_{\mathrm{B}}T}{\pi m}\ . \tag{5.44}$$

The assumption that $S_{f_{\mathrm{N}}}(\omega)$ is a smooth function of ω on the scale γ, together with fact that the dominant effect of the fluctuating force $S_{f_{\mathrm{N}}}(\omega)$ comes from the region near $\omega = 0$, allows employing the approximation $S_{f_{\mathrm{N}}}(\omega) = S_{f_{\mathrm{N}}}(0)$, i.e.

$$S_{f_{\mathrm{N}}}(\omega) = \frac{2\gamma k_{\mathrm{B}}T}{\pi m}\ . \tag{5.45}$$

This implies according to Eq. (5.41) that

$$S_A(\omega) = \frac{2\gamma k_{\mathrm{B}}T}{\pi m} \frac{1}{\gamma^2+\omega^2}\ . \tag{5.46}$$

With the help of Eq. (5.14), which relates the frequency correlation function $\langle f_{\mathrm{N}}^*(\omega') f_{\mathrm{N}}(\omega)\rangle$ of the fluctuating force $S_{f_{\mathrm{N}}}(t)$ with its power spectrum $S_{f_{\mathrm{N}}}(\omega)$, one finds that [see Eq. (5.45)]

$$\begin{aligned}\langle f_{\mathrm{N}}^*(\omega') f_{\mathrm{N}}(\omega)\rangle &= 2\pi S_{f_{\mathrm{N}}}(\omega)\,\delta(\omega-\omega') \\ &= \frac{4\gamma k_{\mathrm{B}}T}{m}\delta(\omega-\omega')\ .\end{aligned} \tag{5.47}$$

Furthermore, the following is expected to hold

$$\langle f_{\mathrm{N}}(\omega') f_{\mathrm{N}}(\omega)\rangle = \langle f_{\mathrm{N}}^{*}(\omega') f_{\mathrm{N}}^{*}(\omega)\rangle = 0 \; . \tag{5.48}$$

To see why Eq. (5.48) is valid, recall that the actual force acting on the mechanical resonator $f(t)$ is given by $f(t) = f_{\mathrm{N}}(t) e^{i\omega_0 t}$ [see Eq. (5.31)]. The assumption that $f(t)$ is stationary implies that all statistical properties of $f(t)$ remain unchanged when $f(t)$ is replaced by $f(t-t_0)$, where t_0 is a constant (i.e. when the signal is shifted in time). Under such transformation $f_{\mathrm{N}}(t) = f(t) e^{-i\omega_0 t}$ is replaced by $f(t-t_0) e^{-i\omega_0 t} = f_{\mathrm{N}}(t') e^{-i\omega_0 t_0}$, where $t' = t - t_0$. The assumption that all statistical properties are independent on t_0 implies that Eq. (5.48) must hold.

5.2.2 Monochromatic Forcing

Consider the case where the resonator is driven by an externally applied force, which is assumed to be given by $Fe^{i\omega_{\mathrm{p}}t} + F^{*}e^{-i\omega_{\mathrm{p}}t}$, where F is a complex constant. For this case Eq. (5.31) becomes

$$\ddot{x} + 2\gamma\dot{x} + \omega_0^2 x = Fe^{i\omega_{\mathrm{p}}t} + F^{*}e^{-i\omega_{\mathrm{p}}t} + f_{\mathrm{N}}(t) e^{i\omega_0 t} \; . \tag{5.49}$$

As in the previous section, a transformation into a rotating frame is performed, however this time the angular frequency is chosen to be ω_{p}, i.e. the variable $a = \dot{x} - \Gamma^{*}x$ is taken to be related to the variable A by the relation $a = Ae^{i\omega_{\mathrm{p}}t}$. For the case of high quality factor (i.e. the case where $\gamma \ll \omega_0$) one has $\Gamma \simeq -\gamma + i\omega_0$. The equation of motion for A is given by [see Eq. (3.41)]

$$\dot{A} + \left[i(\omega_{\mathrm{p}} - \omega_0) + \gamma\right] A = F + F^{*}e^{-2i\omega_{\mathrm{p}}t} + f_{\mathrm{N,d}}(t) \; , \tag{5.50}$$

where

$$f_{\mathrm{N,d}}(t) = f_{\mathrm{N}}(t) e^{i(\omega_0 - \omega_{\mathrm{p}})t} \; . \tag{5.51}$$

In what follows the case where $|\omega_{\mathrm{p}} - \omega_0| \ll \omega_{\mathrm{p}}$ is assumed, i.e. it is assumed that the driving frequency is close to resonance. In the RWA the term $F^{*}e^{-2i\omega_{\mathrm{p}}t}$, which oscillates at angular frequency $2\omega_{\mathrm{p}}$, is disregarded, and consequently the equation of motion for A becomes

$$\dot{A} + \left[i(\omega_{\mathrm{p}} - \omega_0) + \gamma\right] A = F + f_{\mathrm{N,d}}(t) \; . \tag{5.52}$$

Exercise 5.2.2. Show that

$$\left\langle f_{\mathrm{N,d}}^{*}(\omega') f_{\mathrm{N,d}}(\omega)\right\rangle = \frac{4\gamma k_{\mathrm{B}}T}{m}\delta(\omega - \omega') \; , \tag{5.53}$$

and

$$\langle f_{\mathrm{N,d}}(\omega') f_{\mathrm{N,d}}(\omega)\rangle = \left\langle f_{\mathrm{N,d}}^{*}(\omega') f_{\mathrm{N,d}}^{*}(\omega)\right\rangle = 0 \; . \tag{5.54}$$

Solution 5.2.2. As can be easily seen from Eq. (5.51) [see also Eqs. (5.3) and (5.38)] the power spectrum of the 'detuned' fluctuating force $f_{\mathrm{N,d}}(t)$ is related to the power spectrum of the fluctuating force $f_{\mathrm{N}}(t)$ by the following relation

$$S_{f_{\mathrm{N,d}}}(\omega) = S_{f_{\mathrm{N}}}(\omega + \omega_0 - \omega_{\mathrm{p}}) \ . \tag{5.55}$$

The assumption that $|\omega_{\mathrm{p}} - \omega_0| \ll \omega_{\mathrm{p}}$ together with the assumption that $S_{f_{\mathrm{N}}}(\omega)$ is a smooth function of ω implies that the approximation $S_{f_{\mathrm{N,d}}}(\omega) = S_{f_{\mathrm{N}}}(\omega)$ can be made, and therefore the power spectrum $S_{f_{\mathrm{N,d}}}(\omega)$ is approximately given by [see Eq. (5.45)]

$$S_{f_{\mathrm{N,d}}}(\omega) = \frac{2\gamma k_{\mathrm{B}} T}{\pi m} \ . \tag{5.56}$$

Furthermore, the frequency correlation functions (5.47) and (5.48) that where derived for f_{N} are expected to hold as well for $f_{\mathrm{N,d}}$

$$\left\langle f^*_{\mathrm{N,d}}(\omega') f_{\mathrm{N,d}}(\omega) \right\rangle = \frac{4\gamma k_{\mathrm{B}} T}{m} \delta(\omega - \omega') \ , \tag{5.57}$$

and

$$\left\langle f_{\mathrm{N,d}}(\omega') f_{\mathrm{N,d}}(\omega) \right\rangle = \left\langle f^*_{\mathrm{N,d}}(\omega') f^*_{\mathrm{N,d}}(\omega) \right\rangle = 0 \ . \tag{5.58}$$

Consider a solution having the form

$$A(t) = A_0 + A_{\mathrm{N}}(t) \ , \tag{5.59}$$

where the constant A_0 is chosen to be given by

$$A_0 = \frac{F}{i(\omega_{\mathrm{p}} - \omega_0) + \gamma} \ . \tag{5.60}$$

Substituting into Eq. (5.52) yields the following equation of motion for $A_{\mathrm{N}}(t)$

$$\dot{A}_{\mathrm{N}} + [i(\omega_{\mathrm{p}} - \omega_0) + \gamma] A_{\mathrm{N}} = f_{\mathrm{N,d}}(t) \ . \tag{5.61}$$

By using the last result (5.61) one finds that the FT $A_{\mathrm{N}\tau}(\omega)$ of the sampling $A_{\mathrm{N}\tau}(t)$ is related to the FT $f_{\mathrm{N,d}\tau}(\omega)$ of the sampling $f_{\mathrm{N,d}}(t)$ by

$$A_{\mathrm{N}\tau}(\omega) = R_{\mathrm{d}}(\omega) f_{\mathrm{N,d}\tau}(\omega) \ , \tag{5.62}$$

where the frequency response function $R_{\mathrm{d}}(\omega)$ is given by

$$R_{\mathrm{d}}(\omega) = \frac{1}{-i(\omega + \omega_0 - \omega_{\mathrm{p}}) + \gamma} \ . \tag{5.63}$$

Thus the power spectrum $S_{A_{\mathrm{N}}}(\omega)$ of $A_{\mathrm{N}}(t)$ and the power spectrum of the fluctuating force $f_{\mathrm{N,d}}(t)$ (which is labeled by $S_{f_{\mathrm{N,d}}}(\omega)$) are related by [see Eq. (5.41)]

$$S_A(\omega) = |R_{\mathrm{d}}(\omega)|^2 S_{f_{\mathrm{N,d}}}(\omega) \ , \tag{5.64}$$

where $S_{f_{\mathrm{N}}}(\omega)$ is given by Eq. (5.56) and where

$$|R_{\mathrm{d}}(\omega)|^2 = \frac{1}{(\omega + \omega_0 - \omega_{\mathrm{p}})^2 + \gamma^2} \ . \tag{5.65}$$

5.2.3 Homodyne Detection

Consider the case where the displacement $x(t)$ is continuously monitored. Using the relations $a = \dot{x} - \Gamma^* x$, $a = Ae^{i\omega_{\mathrm{p}}t}$ and assuming $\gamma \ll \omega_0$ one finds that $x(t)$ can be expressed in terms of $A(t)$ as [see Eq. (3.46]

$$x(t) = \frac{a - a^*}{2i\omega_0} = \frac{Ae^{i\omega_{\mathrm{p}}t} - A^* e^{-i\omega_{\mathrm{p}}t}}{2i\omega_0} \ . \tag{5.66}$$

In homodyne detection $x(t)$ is mixed (i.e. multiplied) with a signal $I(t)$ that oscillates at the same frequency ω_{p} as the drive with a fixed real phase ϕ_{LO} and a fixed real amplitude I_0

$$I(t) = 2I_0 \operatorname{Re}\left(e^{i(\omega_{\mathrm{p}}t + \phi_{\mathrm{LO}})}\right) \ . \tag{5.67}$$

The normalized output of the mixer, which is labeled as

$$X_{\phi_{\mathrm{LO}}}(t) = \frac{2\omega_0}{I_0} x(t) I(t) \ , \tag{5.68}$$

is given by

$$X_{\phi_{\mathrm{LO}}}(t) = \left(-iAe^{i\omega_{\mathrm{p}}t} + iA^* e^{-i\omega_{\mathrm{p}}t}\right)\left(e^{i(\omega_{\mathrm{p}}t + \phi_{\mathrm{LO}})} + e^{-i(\omega_{\mathrm{p}}t + \phi_{\mathrm{LO}})}\right) \ . \tag{5.69}$$

Recall that the envelope function $A(t) = A_0 + A_{\mathrm{N}}(t)$ [see Eq. (5.59)] is assumed to vary slowly in time, i.e. the change in $A(t)$ over time period of the order of ω_{p}^{-1} is assumed to be small. Thus the spectrum of $X_{\phi_{\mathrm{LO}}}(t)$ is expected to contain slowly varying terms and terms that oscillate at frequency close to $2\omega_{\mathrm{p}}$. Consider the case where a low pass filter is used at the output of the mixer to eliminate the fast terms that oscillate at frequency $2\omega_{\mathrm{p}}$. In that case the output signal is taken to be given by

$$X_{\phi_{\mathrm{LO}}}(t) = X_0 + X_{\mathrm{N}}(t) \ , \tag{5.70}$$

where

$$X_0 = -iA_0 e^{-i\phi_{\mathrm{LO}}} + iA_0^* e^{i\phi_{\mathrm{LO}}} \ , \tag{5.71}$$

and

$$X_{\mathrm{N}}(t) = -iA_{\mathrm{N}}(t)\, e^{-i\phi_{\mathrm{LO}}} + iA_{\mathrm{N}}^{*}(t)\, e^{i\phi_{\mathrm{LO}}} \ . \tag{5.72}$$

Consider a measurement in which $X_{\phi_{\mathrm{LO}}}(t)$ is continuously monitored in the time interval $(-\tau/2, \tau/2)$. Let $\hat{X}_0$ be an estimator of the average value of $X_{\phi_{\mathrm{LO}}}(t)$, which is taken to be given by

$$\hat{X}_0 = \frac{1}{\tau}\int_0^{\tau} \mathrm{d}t\ X_{\phi_{\mathrm{LO}}}(t) \ . \tag{5.73}$$

Exercise 5.2.3. Show that the variance $\left(\hat{X}_0 - \left\langle \hat{X}_0 \right\rangle\right)^2$ of the random variable $\hat{X}_0$ is given by

$$\left(\hat{X}_0 - \left\langle \hat{X}_0 \right\rangle\right)^2 = \frac{8\gamma k_{\mathrm{B}} T}{m\tau} \frac{1}{\left(\omega_{\mathrm{p}} - \omega_0\right)^2 + \gamma^2} \ . \tag{5.74}$$

Solution 5.2.3. The expectation value of $X_{\mathrm{N}}(t)$ vanishes since $\langle A_{\mathrm{N}}(t)\rangle = 0$, and therefore the expectation value of $\hat{X}_0$ is given by

$$\left\langle \hat{X}_0 \right\rangle = X_0 \ . \tag{5.75}$$

The variance $\left(\hat{X}_0 - \left\langle \hat{X}_0 \right\rangle\right)^2$ of the random variable $\hat{X}_0$ depends on the sampling time τ. For relatively long sampling times (i.e. in the limit $\tau \to \infty$) the variance is given according to Eq. (5.22) by

$$\left(\hat{X}_0 - \left\langle \hat{X}_0 \right\rangle\right)^2 = \frac{2\pi}{\tau} S_{X_{\mathrm{N}}}(0) \ . \tag{5.76}$$

The last result is valid provided that τ is much longer than the so-called correlation time of the randomly fluctuating signal $X_{\mathrm{N}}(t)$. Using the expression for $S_{X_{\mathrm{N}}}(\omega)$ (5.78), which is derived below, one finds that the variance is given by

$$\left(\hat{X}_0 - \left\langle \hat{X}_0 \right\rangle\right)^2 = \frac{8\gamma k_{\mathrm{B}} T}{m\tau} \frac{1}{\left(\omega_{\mathrm{p}} - \omega_0\right)^2 + \gamma^2} \ . \tag{5.77}$$

Exercise 5.2.4. Show that

$$S_{X_{\mathrm{N}}}(\omega) = \frac{4\gamma k_{\mathrm{B}} T}{\pi m} \frac{1}{\left(\omega_{\mathrm{p}} - \omega_0 - \omega\right)^2 + \gamma^2} \ . \tag{5.78}$$

Solution 5.2.4. According to Eq. (5.14) the following holds

$$\left\langle X_{\mathrm{N}}^{*}(\omega')\, X_{\mathrm{N}}(\omega)\right\rangle = 2\pi S_{X_{\mathrm{N}}}(\omega)\, \delta(\omega - \omega') \ , \tag{5.79}$$

where [see Eq. (5.72)]

$$X_{\mathrm{N}}(\omega) = -iA_{\mathrm{N}}(\omega)\, e^{-i\phi_{\mathrm{LO}}} + iA_{\mathrm{N}}^{*}(\omega)\, e^{i\phi_{\mathrm{LO}}} \ . \tag{5.80}$$

With the help of Eq. (5.62)one finds that

$$
\begin{aligned}
&\left\langle X_{\mathrm{N}}^{*}(\omega') X_{\mathrm{N}}(\omega)\right\rangle \\
&=\left\langle\left(iA_{\mathrm{N}}^{*}(\omega') e^{i\phi_{\mathrm{LO}}}-iA_{\mathrm{N}}(\omega') e^{-i\phi_{\mathrm{LO}}}\right)\left(-iA_{\mathrm{N}}(\omega) e^{-i\phi_{\mathrm{LO}}}+iA_{\mathrm{N}}^{*}(\omega) e^{i\phi_{\mathrm{LO}}}\right)\right\rangle \\
&=\left\langle A_{\mathrm{N}}^{*}(\omega') A_{\mathrm{N}}(\omega)\right\rangle+\left\langle A_{\mathrm{N}}(\omega') A_{\mathrm{N}}^{*}(\omega)\right\rangle \\
&\quad-\left\langle A_{\mathrm{N}}^{*}(\omega') A_{\mathrm{N}}^{*}(\omega)\right\rangle e^{2i\phi_{\mathrm{LO}}}-\left\langle A_{\mathrm{N}}(\omega') A_{\mathrm{N}}(\omega)\right\rangle e^{-2i\phi_{\mathrm{LO}}} \\
&=R_{\mathrm{d}}(\omega) R_{\mathrm{d}}^{*}(\omega')\left\langle f_{\mathrm{N,d}}^{*}(\omega') f_{\mathrm{N,d}}(\omega)\right\rangle \\
&\quad+R_{\mathrm{d}}(\omega') R_{\mathrm{d}}^{*}(\omega)\left\langle f_{\mathrm{N,d}}(\omega') f_{\mathrm{N,d}}^{*}(\omega)\right\rangle \\
&\quad-R_{\mathrm{d}}^{*}(\omega') R_{\mathrm{d}}^{*}(\omega)\left\langle f_{\mathrm{N,d}}^{*}(\omega') f_{\mathrm{N,d}}^{*}(\omega)\right\rangle e^{2i\phi_{\mathrm{LO}}} \\
&\quad-R_{\mathrm{d}}(\omega') R_{\mathrm{d}}(\omega)\left\langle f_{\mathrm{N,d}}(\omega') f_{\mathrm{N,d}}(\omega)\right\rangle e^{-2i\phi_{\mathrm{LO}}}\ ,
\end{aligned}
\tag{5.81}
$$

where the frequency response function $R_{\mathrm{d}}(\omega)$ is given by

$$
R_{\mathrm{d}}(\omega)=\frac{1}{-i(\omega+\omega_0-\omega_{\mathrm{p}})+\gamma}\ , \tag{5.82}
$$

thus according to Eqs. (5.57) and (5.58) one has

$$
\left\langle X_{\mathrm{N}}^{*}(\omega') X_{\mathrm{N}}(\omega)\right\rangle=\frac{8\gamma k_{\mathrm{B}}T\left|R_{\mathrm{d}}(\omega)\right|^2}{m}\delta(\omega-\omega')\ . \tag{5.83}
$$

The last result together with Eq. (5.79) implies that

$$
S_{X_{\mathrm{N}}}(\omega)=\frac{4\gamma k_{\mathrm{B}}T\left|R_{\mathrm{d}}(\omega)\right|^2}{\pi m}=\frac{4\gamma k_{\mathrm{B}}T}{\pi m}\frac{1}{(\omega+\omega_0-\omega_{\mathrm{p}})^2+\gamma^2}\ . \tag{5.84}
$$

5.3 Responsivity

The responsivity factor $\mathcal{R}_0$, which is defined by

$$
\mathcal{R}_0=\left|\frac{\partial X_0}{\partial\omega_0}\right|\ , \tag{5.85}
$$

represents the dependence of the average value of the measured signal X_0 on the resonance frequency ω_0.

Exercise 5.3.1. Show that for the case of homodyne detection the following holds

$$
\mathcal{R}_0=2\left|F\right|\frac{\left|\cos(\phi_{\mathrm{F}}-\phi_{\mathrm{LO}}-2\phi_{\mathrm{d}})\right|}{(\omega_{\mathrm{p}}-\omega_0)^2+\gamma^2}\ , \tag{5.86}
$$

where ϕ_{F} is the phase of the complex force amplitude F, i.e. $F=\left|F\right|e^{i\phi_{\mathrm{F}}}$, and where

$$
\phi_{\mathrm{d}}=\tan^{-1}\frac{\omega_{\mathrm{p}}-\omega_0}{\gamma}\ . \tag{5.87}
$$

Solution 5.3.1. With the help of Eqs. (5.60) and (5.71) one finds that

$$\begin{aligned} X_0 &= -iA_0 e^{-i\phi_{\mathrm{LO}}} + iA_0^* e^{i\phi_{\mathrm{LO}}} \\ &= \frac{|F| e^{i\left(\phi_{\mathrm{F}} - \phi_{\mathrm{LO}} - \frac{\pi}{2}\right)}}{i\left(\omega_{\mathrm{p}} - \omega_0\right) + \gamma} + \frac{|F| e^{-i\left(\phi_{\mathrm{F}} - \phi_{\mathrm{LO}} - \frac{\pi}{2}\right)}}{-i\left(\omega_{\mathrm{p}} - \omega_0\right) + \gamma} \\ &= 2|F| \operatorname{Re}\left(\frac{e^{i\left(\phi_{\mathrm{F}} - \phi_{\mathrm{LO}} - \frac{\pi}{2}\right)}}{i\left(\omega_{\mathrm{p}} - \omega_0\right) + \gamma}\right) , \end{aligned} \tag{5.88}$$

where $F = |F| e^{i\phi_{\mathrm{F}}}$, thus

$$\frac{\partial X_0}{\partial \omega_0} = 2|F| \operatorname{Re}\left(\frac{e^{i(\phi_{\mathrm{F}} - \phi_{\mathrm{LO}})}}{\left(i\left(\omega_{\mathrm{p}} - \omega_0\right) + \gamma\right)^2}\right) . \tag{5.89}$$

By using the identity [see Eq. (3.48)]

$$\frac{1}{i\left(\omega_{\mathrm{p}} - \omega_0\right) + \gamma} = \frac{e^{-i\phi_{\mathrm{d}}}}{\sqrt{\left(\omega_{\mathrm{p}} - \omega_0\right)^2 + \gamma^2}} \tag{5.90}$$

where

$$\phi_{\mathrm{d}} = \tan^{-1} \frac{\omega_{\mathrm{p}} - \omega_0}{\gamma} , \tag{5.91}$$

one finds that

$$\mathcal{R}_0 = \left|\frac{\partial X_0}{\partial \omega_0}\right| = 2|F| \frac{\left|\cos\left(\phi_{\mathrm{F}} - \phi_{\mathrm{LO}} - 2\phi_{\mathrm{d}}\right)\right|}{\left(\omega_{\mathrm{p}} - \omega_0\right)^2 + \gamma^2} . \tag{5.92}$$

5.4 Figures of Merit

The homodyne detection signal $X_{\phi_{\mathrm{LO}}}(t)$ allows monitoring changes in the resonance frequency ω_0. The responsivity factor $\mathcal{R}_0 = |\partial X_0 / \partial \omega_0|$ (5.92) characterizes the change in X_0 per a given change in ω_0. In resonant detection the measured physical parameter of interest, which is denoted as $\mathcal{P}$, is coupled to the resonator in such a way that ω_0 becomes effectively $\mathcal{P}$ dependent, i.e. $\omega_0 = \omega_0(\mathcal{P})$. This dependency is characterized by the responsivity factor

$$\mathcal{R} = \left|\frac{\partial X_0}{\partial \mathcal{P}}\right| = \mathcal{R}_0 \left|\frac{\partial \omega_0}{\partial \mathcal{P}}\right| . \tag{5.93}$$

In general, any detection scheme employed for monitoring the parameter of interest $\mathcal{P}$ can be characterized by two important figures of merit. The first is the minimum detectable change in $\mathcal{P}$, denoted as $\delta\mathcal{P}$. This parameter is determined by the above mentioned responsivity factor $\mathcal{R}$, the noise level, which is usually characterized by the power spectrum of the measured output signal ($X_{\phi_{\mathrm{LO}}}(t)$ for the case of homodyne detection), and by the averaging time τ that is employed for measuring the output signal. The second figure of merit is the ring-down time t_{RD}, which is a measure of the detector's response time to a sudden change in $\mathcal{P}$.

Minimum Detectable Change. Consider a measurement in which $X_{\phi_{\mathrm{LO}}}(t)$ is continuously monitored in the time interval $(-\tau/2, \tau/2)$. The estimator $\hat{X}_0$ [see Eq. (5.73)] has an expectation value X_0 [see Eq. (5.71)] and variance $\left(\hat{X}_0 - \left\langle \hat{X}_0 \right\rangle\right)^2$ [see Eq. (5.77)]. The minimum detectable change $\delta\mathcal{P}$, which characterizes the resolution in determining $\mathcal{P}$ based on the estimator $\hat{X}_0$, is thus given by

$$\delta\mathcal{P} = \delta X_0 \mathcal{R}^{-1} = \sqrt{\left(\hat{X}_0 - \left\langle \hat{X}_0 \right\rangle\right)^2} \mathcal{R}_0^{-1} \left| \frac{\partial \omega_0}{\partial \mathcal{P}} \right|^{-1} . \tag{5.94}$$

With the help of Eqs. (5.77) and (5.92) this becomes

$$\delta\mathcal{P} = \frac{\sqrt{\frac{8\gamma k_{\mathrm{B}} T}{m\tau}} \left| \frac{\partial \omega_0}{\partial \mathcal{P}} \right|^{-1}}{2 \left| A_0 \cos\left(\phi_{\mathrm{F}} - \phi_{\mathrm{LO}} - 2\phi_{\mathrm{d}}\right) \right|} . \tag{5.95}$$

where [see Eq. (5.60)]

$$|A_0| = \frac{|F|}{\sqrt{\left(\omega_{\mathrm{p}} - \omega_0\right)^2 + \gamma^2}} . \tag{5.96}$$

When the phase of the local oscillator ϕ_{LO} is chosen such that $\left|\cos\left(\phi_{\mathrm{F}} - \phi_{\mathrm{LO}} - 2\phi_{\mathrm{d}}\right)\right| = 1$ in order to minimize $\delta\mathcal{P}$ the minimum detectable change becomes

$$\delta\mathcal{P} = \frac{\sqrt{\frac{8\gamma k_{\mathrm{B}} T}{m\tau}} \left| \frac{\partial \omega_0}{\partial \mathcal{P}} \right|^{-1}}{2 |A_0|} . \tag{5.97}$$

Recall that the total energy that is stored in the resonator, which is labeled by U_0, is given by $(m/2) |A_0|^2$ [see Eq. (5.33)]. Thus in terms of the stored energy U_0 one has

$$\delta\mathcal{P} = \sqrt{\frac{\gamma k_{\mathrm{B}} T}{U_0 \tau}} \left| \frac{\partial \omega_0}{\partial \mathcal{P}} \right|^{-1} . \tag{5.98}$$

Ring Down Time. The ring-down time t_{RD} characterizes the detector's response time to a sudden change in $\mathcal{P}$, i.e. the time it takes to approach steady state. This time scale is on the order of the inverse damping rate, i.e. $t_{\mathrm{RD}} \simeq \gamma^{-1}$. As can be seen from Eq. (5.98), sensitivity enhancement (i.e. reduction in $\delta\mathcal{P}$) can be achieved by increasing the quality factor Q (i.e. reducing γ), however, this unavoidably will be accompanied by an undesirable increase in the ring-down time, namely, slowing down the response of the system to changes in $\mathcal{P}$.

5.5 Problems

1. Let $z(t)$ be a real stationary random signal. The quantity $P_z(\omega)$ is defined by

$$P_z(\omega) = \frac{1}{2\pi} \lim_{\tau\rightarrow\infty} \frac{1}{\tau} \left[\left(\int_{-\tau/2}^{\tau/2} \mathrm{d}t \; z(t) \cos(\omega t) \right)^2 + \left(\int_{-\tau/2}^{\tau/2} \mathrm{d}t \; z(t) \sin(\omega t) \right)^2 \right] . \tag{5.99}$$

   Show that $P_z(\omega)$ is the power spectrum of $z(t)$.
2. Consider a mechanical resonator with mass m, resonance frequency ω_0, and damping rate γ. The system is in thermal equilibrium at temperature T. Calculate the power spectrum $S_x(\omega)$ of x.
3. **Nyquist's noise** - Consider the circuit shown in Fig. 5.1, which consists of a capacitor having capacitance C, an inductor having inductance L, and a resistor having resistance R, all serially connected. The system is assumed to be in thermal equilibrium at temperature T. To model the effect of thermal fluctuations a fictitious voltage source is added, which produces a random fluctuating voltage $V(t)$. Find the spectral densities $S_q(\omega)$ and $S_V(\omega)$ of the charge on the capacitor and of the voltage source respectively.

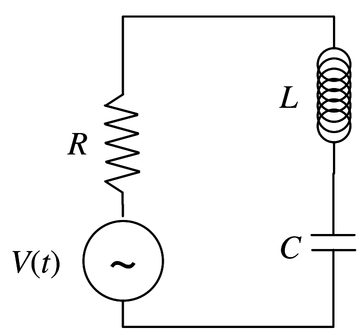


Fig. 5.1. LCR resonator.

4. **Phase noise and jitter -** Consider the signal

$$z(t) = A \sin(\omega_0 t + \phi_\mathrm{n}(t)) \; , \tag{5.100}$$

   where A and ω_0 are both real constants and the phase $\phi_\mathrm{n}(t)$ is allowed to vary in time. The following is assumed to hold

$$\langle \cos(\phi_\mathrm{n}(t+t') - \phi_\mathrm{n}(t)) \rangle = \exp\left(-\gamma_\mathrm{g} |t'|\right) \; , \tag{5.101}$$

where γ_{g} is the so-called jitter rate. Show that when $\gamma_{\mathrm{g}} \ll \omega_0$ the power spectrum $P_z(\omega)$ of $z(t)$ is given by

$$P_z(\omega) = \frac{A^2}{4\pi} \frac{\gamma_{\mathrm{g}}}{\gamma_{\mathrm{g}}^2 + (\omega - \omega_0)^2} . \tag{5.102}$$

5. **Langevin equation** - Consider the Langevin equation for the variable ϕ

$$\dot{\phi} + \Gamma\phi = \vartheta , \tag{5.103}$$

where Γ is a constant, $\vartheta(t)$ represents a white noise

$$\langle \vartheta(t) \vartheta(t') \rangle = 2\Theta\delta(t - t') , \tag{5.104}$$

and Θ is a constant.

a) Show that to lowest nonvanishing order in time t the following holds

$$\left\langle e^{i(\phi(t)-\phi(0))} \right\rangle = 1 - \Theta t + O\left(t^2\right) . \tag{5.105}$$

Note that the above result is independent on Γ.

b) For the case where $\Gamma > 0$, in steady state show that the following holds

$$\left\langle \phi^2 \right\rangle = \frac{\Theta}{\Gamma} . \tag{5.106}$$

5.6 Solutions

1. The following holds

$$\begin{aligned}
&\left(\int_{-\tau/2}^{\tau/2} \mathrm{d}t \; z(t) \cos(\omega t) \right)^2 + \left(\int_{-\tau/2}^{\tau/2} \mathrm{d}t \; z(t) \sin(\omega t) \right)^2 \\
&= \int_{-\tau/2}^{\tau/2} \mathrm{d}t \int_{-\tau/2}^{\tau/2} \mathrm{d}t' \; z(t) z(t') \cos(\omega(t - t')) ,
\end{aligned} \tag{5.107}$$

thus in terms of the sampling function $z_\tau(t)$, which is defined by [see Eq. (5.1)]

$$z_\tau(t) = \begin{cases} z(t) & -\tau/2 < t < \tau/2 \\ 0 & \text{else} \end{cases} , \tag{5.108}$$

one finds that

$$P_z(\omega) = \frac{1}{2\pi} \lim_{\tau\to\infty} \frac{1}{\tau} \int_{-\infty}^{\infty} \mathrm{d}t \int_{-\infty}^{\infty} \mathrm{d}t' \; z_\tau(t)\, z_\tau(t') \cos(\omega(t-t')) \; . \quad (5.109)$$

The variable transformation $t'' = t - t'$ leads to

$$\begin{aligned} P_z(\omega) &= \frac{1}{2\pi} \lim_{\tau\to\infty} \frac{1}{\tau} \int_{-\infty}^{\infty} \mathrm{d}t' \int_{-\infty}^{\infty} \mathrm{d}t'' \; z_\tau(t'+t'')\, z_\tau(t') \cos(\omega t'') \\ &= \frac{1}{2\pi} \int_{-\infty}^{\infty} \mathrm{d}t'' \; \cos(\omega t'') \lim_{\tau\to\infty} \frac{1}{\tau} \int_{-\infty}^{\infty} \mathrm{d}t' z_\tau(t'+t'')\, z_\tau(t') \; , \end{aligned} \quad (5.110)$$

thus in terms of the autocorrelation function $C_z(t)$ [see Eq. (5.9)] one finds that

$$P_z(\omega) = \frac{1}{2\pi} \int_{-\infty}^{\infty} \mathrm{d}t'' \; \cos(\omega t'')\, C_z(t'') \; . \quad (5.111)$$

According to the Wiener-Khinchine Theorem [see Eq. (5.10)] the following holds

$$C_z(t'') = \int_{-\infty}^{\infty} \mathrm{d}\omega' \; e^{i\omega' t''} S_z(\omega') \; , \quad (5.112)$$

where $S_z(\omega)$ is the power spectrum of $z(t)$, thus

$$\begin{aligned} P_z(\omega) &= \frac{1}{2\pi} \int_{-\infty}^{\infty} \mathrm{d}t'' \; \cos(\omega t'') \int_{-\infty}^{\infty} \mathrm{d}\omega' \; e^{i\omega' t''} S_z(\omega') \\ &= \int_{-\infty}^{\infty} \mathrm{d}\omega' \; S_z(\omega') \underbrace{\frac{1}{2\pi} \int_{-\infty}^{\infty} \mathrm{d}t'' \frac{e^{i(\omega+\omega')t''} + e^{-i(\omega-\omega')t''}}{2}}_{\delta(\omega+\omega')+\delta(\omega-\omega')} \\ &= \frac{S_z(-\omega) + S_z(\omega)}{2} \; . \end{aligned} \quad (5.113)$$

Using the fact that $z(t)$ is real one finds that $z_\tau(-\omega) = z_\tau^*(\omega)$ [see Eq. (5.2)] and consequently [see Eq. (5.3)] $S_z(-\omega) = S_z(\omega)$, thus $P_z(\omega) = S_z(\omega)$.

2. Let $x_\tau(t)$ be sampling of the displacement $x(t)$ in the time interval $(-\tau/2, \tau/2)$. The following holds

$$x_\tau(t) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} \mathrm{d}\omega\ x_\tau(\omega)\, e^{-i\omega t} \ , \tag{5.114}$$

$$\dot{x}_\tau(t) = \frac{-i\omega}{\sqrt{2\pi}} \int_{-\infty}^{\infty} \mathrm{d}\omega\ x_\tau(\omega)\, e^{-i\omega t} \ , \tag{5.115}$$

$$\ddot{x}_\tau(t) = \frac{(-i\omega)^2}{\sqrt{2\pi}} \int_{-\infty}^{\infty} \mathrm{d}\omega\ x_\tau(\omega)\, e^{-i\omega t} \ , \tag{5.116}$$

thus the equation of motion (5.31), which is given by

$$\ddot{x} + 2\gamma\dot{x} + \omega_0^2 x = f_{\mathrm{N}}(t)\, e^{i\omega_0 t} \ , \tag{5.117}$$

yields

$$x_\tau(\omega) = \frac{f_{\mathrm{N}\tau}(\omega + \omega_0)}{\omega_0^2 - \omega^2 - 2i\gamma\omega} \ , \tag{5.118}$$

where $f_{\mathrm{N}\tau}(t)$ is sampling of the fluctuating force $f_{\mathrm{N}}(t)$. With the help of Eqs. (5.14), (5.47) and (5.48) one finds that

$$S_x(\omega) = \frac{4\gamma k_{\mathrm{B}} T}{\pi m} \frac{1}{\left(\omega_0^2 - \omega^2\right)^2 + 4\gamma^2\omega^2} \ . \tag{5.119}$$

When $|\omega - \omega_0| \ll \omega_0$ it is convenient to employ the near resonance approximation (3.62), which leads to

$$S_x(\omega) = \frac{\gamma k_{\mathrm{B}} T}{\pi m \omega_0^2} \frac{1}{\left(\omega - \omega_0\right)^2 + \gamma^2} \ . \tag{5.120}$$

Note that, as is expected from the equipartition theorem, according to Eq. (5.5) the following holds

$$\left\langle x^2 \right\rangle = \int_{-\infty}^{\infty} \mathrm{d}\omega\ S_x(\omega) = \frac{k_{\mathrm{B}} T}{m\omega_0^2} \ . \tag{5.121}$$

3. Let $q(t)$ be the charge stored in the capacitor at time t. The classical equation of motion, which is given by

$$\frac{q}{C} + L\ddot{q} + R\dot{q} = V(t) \ , \tag{5.122}$$

represents Kirchhoff's voltage law. The energy stored in the capacitor is given by $q^2/2C$, thus according to the equipartition theorem in thermal equilibrium the following must hold

$$\frac{\left\langle q^2 \right\rangle}{2C} = \frac{k_{\mathrm{B}} T}{2} \ . \tag{5.123}$$

Comparing with Eq. (5.121) shows that the current problem is equivalent to the problem of a mechanical resonator having effective mass

$m_{\text{eff}} = L$, effective damping rate $\gamma_{\text{eff}} = R/2L$, effective angular resonance frequency $\omega_{0,\text{eff}} = 1/\sqrt{LC}$, driven by an effective fluctuating force given by $f_{\text{N,eff}}(t) = V(t)/L$. Thus Eq. (5.120) yields

$$S_q(\omega) = \frac{RCk_{\text{B}}T}{2\pi L} \frac{1}{\left(\frac{R}{2L}\right)^2 + \left(\omega - \frac{1}{LC}\right)^2} , \tag{5.124}$$

and Eq. (5.56) yields

$$S_V(\omega) = \frac{2Rk_{\text{B}}T}{\pi} . \tag{5.125}$$

4. The correlation function $C_z(t')$ is given by [see Eq. (5.9)]

$$C_z(t') = A^2 \langle \sin(\omega_0 (t+t') + \phi_{\text{n}}(t+t')) \sin(\omega_0 t + \phi_{\text{n}}(t)) \rangle . \tag{5.126}$$

The following holds

$$\begin{aligned}
&\sin(\omega_0 (t+t') + \phi_{\text{n}}(t+t')) \sin(\omega_0 t + \phi_{\text{n}}(t)) \\
&= \frac{\cos(\omega_0 t') \cos(\phi_{\text{n}}(t+t') - \phi_{\text{n}}(t))}{2} \\
&- \frac{\sin(\omega_0 t') \sin(\phi_{\text{n}}(t+t') - \phi_{\text{n}}(t))}{2} \\
&- \frac{\cos(\omega_0 (2t+t')) \cos(\phi_{\text{n}}(t+t') + \phi_{\text{n}}(t))}{2} \\
&+ \frac{\sin(\omega_0 (2t+t')) \sin(\phi_{\text{n}}(t+t') + \phi_{\text{n}}(t))}{2} .
\end{aligned} \tag{5.127}$$

Only the first term contributes to the expectation value

$$C_z(t') = \frac{A^2 \cos(\omega_0 t')}{2} \langle \cos(\phi_{\text{n}}(t+t') - \phi_{\text{n}}(t)) \rangle . \tag{5.128}$$

With the help of Eq. (5.101) one finds that the power spectrum is given by [see Eq. (5.111)]

$$\begin{aligned}
P_z(\omega) &= \frac{A^2}{4\pi} \int_{-\infty}^{\infty} \mathrm{d}t' \; \cos(\omega t') \cos(\omega_0 t') \exp(-\gamma_{\text{g}} |t'|) \\
&= \frac{A^2}{4\pi} \left(\frac{\gamma_{\text{g}}}{\gamma_{\text{g}}^2 + (\omega - \omega_0)^2} + \frac{\gamma_{\text{g}}}{\gamma_{\text{g}}^2 + (\omega + \omega_0)^2} \right) .
\end{aligned} \tag{5.129}$$

The assumption $\gamma_{\text{g}} \ll \omega_0$ leads to Eq. (5.102). Note that the following holds [see Eq. (5.101)]

$$\left\langle \left(\phi_{\mathrm{n}}\left(t+t'\right)-\phi_{\mathrm{n}}\left(t\right)\right)^{2}\right\rangle = 2\gamma_{\mathrm{g}}\left|t'\right| + O\left(\left|t'\right|^{2}\right) \ , \tag{5.130}$$

thus in the short time limit $\phi_{\mathrm{n}}(t)$ undergoes a random walk process with a characteristic rate γ_{g}.

5. The solution of Eq. (5.103) is given by

$$\phi(t) = \phi(0)\, e^{-\Gamma t} + \int_0^t \mathrm{d}t'\; e^{-\Gamma\left(t-t'\right)}\vartheta\left(t'\right) \ . \tag{5.131}$$

Without loss of generality, it is assumed that $\phi(0) = 0$.

a) To lowest nonvanishing order the following holds (note that $\langle\phi(t)\rangle = 0$)

$$\left\langle e^{i\phi(t)}\right\rangle = 1 - \frac{\left\langle \phi^{2}(t)\right\rangle}{2} \ , \tag{5.132}$$

where [see Eq. (5.104)]

$$\begin{aligned}\left\langle \phi^{2}(t)\right\rangle &= \int_0^t \mathrm{d}t' \int_0^t \mathrm{d}t''\; e^{-\Gamma\left(2t-t'-t''\right)}\left\langle \vartheta\left(t'\right)\vartheta\left(t''\right)\right\rangle \\ &= 2\Theta \int_0^t \mathrm{d}t'\; e^{-2\Gamma\left(t-t'\right)} \\ &= \Theta\frac{1-e^{-2\Gamma t}}{\Gamma} \\ &= 2\Theta t + O\left(t^{2}\right) \ ,\end{aligned} \tag{5.133}$$

thus Eq. (5.105) holds.

b) For the case where $\Gamma > 0$ Eq. (5.103) has a steady state solution $\langle\phi\rangle = 0$. The Fourier expansions of ϑ and ϕ are given by

$$\vartheta(t) = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} \mathrm{d}\omega\; \vartheta(\omega)\, e^{-i\omega t} \ , \tag{5.134}$$

$$\phi(t) = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} \mathrm{d}\omega\; \phi(\omega)\, e^{-i\omega t} \ . \tag{5.135}$$

With the help of Eq. (5.103) one finds that

$$\phi(\omega) = \frac{\vartheta(\omega)}{-i\omega+\Gamma} \ . \tag{5.136}$$

The above result (5.136) together with Eq. (5.104) and the inverse Fourier transform

$$\vartheta(\omega) = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} \mathrm{d}t\; \vartheta(t)\, e^{i\omega t} \ , \tag{5.137}$$

yields (recall that $\int_{-\infty}^{\infty} \mathrm{d}t\ e^{i(\omega-\omega')t} = 2\pi\delta(\omega-\omega')$) [compare with Eq. (5.14)]

$$
\begin{aligned}
\langle \phi^* (\omega') \phi (\omega) \rangle &= \frac{\langle \vartheta^* (\omega') \vartheta (\omega) \rangle}{(i\omega' + \Gamma)(-i\omega + \Gamma)} \\
&= \frac{\frac{1}{2\pi} \int_{-\infty}^{\infty} \mathrm{d}t' \int_{-\infty}^{\infty} \mathrm{d}t\ e^{i\omega' t'} e^{i\omega t} \langle \vartheta (t') \vartheta (t) \rangle}{(i\omega' + \Gamma)(-i\omega + \Gamma)} \\
&= 2\pi S_\phi (\omega) \delta (\omega - \omega')\ ,
\end{aligned}
\tag{5.138}
$$

where

$$
S_\phi (\omega) = \frac{\Theta}{\pi (\Gamma^2 + \omega^2)}\ . \tag{5.139}
$$

The integrated power spectrum $S_\phi (\omega)$ leads to Eq. (5.106) [compare with Eq. (5.5)]

$$
\begin{aligned}
\langle \phi^2 \rangle &= \int_{-\infty}^{\infty} \mathrm{d}\omega\ S_\phi (\omega) \\
&= \frac{\Theta}{\pi} \int_{-\infty}^{\infty} \frac{\mathrm{d}\omega}{\Gamma^2 + \omega^2} \\
&= \frac{\Theta}{\pi\Gamma} \int_{-\infty}^{\infty} \frac{\mathrm{d}y}{1 + y^2} \\
&= \frac{\Theta}{\Gamma}\ .
\end{aligned}
\tag{5.140}
$$

Note that the same result (5.106) can also be derived using Eq. (5.133).

6. Nonlinear Oscillations

In this chapter two examples of nonlinear oscillations are discussed. In the first one a harmonic oscillator is employed as a parametric amplifier, whereas in the second one a driven Duffing oscillator is analyzed.

6.1 Parametric Amplifier

A mechanical resonator is characterized by its mass m, its angular resonance frequency ω_0 and its damping rate γ. When these parameters are modulated in time the system is said to be parametrically excited. In the example below we consider the case of a simple pendulum whose resonance frequency is periodically modulated in time. The pendulum, which is made of a weight of mass m and a massless string of length l, is placed in a gravitational field having acceleration g (see Fig. 6.1).

Exercise 6.1.1. First consider the case where l is taken to be a constant. Find the equation of motion for the system.

Solution 6.1.1. In terms of the angle θ (see Fig. 6.1) the displacement of the mass is given by $x = l \sin\theta$ and $y = -l\cos\theta$. The kinetic energy is thus given by $m\left(\dot{x}^2 + \dot{y}^2\right)/2 = ml^2\dot{\theta}^2\left(\cos^2\theta + \sin^2\theta\right)/2 = ml^2\dot{\theta}^2/2$, the potential energy by $-mgl\cos\theta$, and the Lagrangian by

$$\mathcal{L} = \frac{ml^2\dot{\theta}^2}{2} + mgl\cos\theta \; . \tag{6.1}$$

The Euler-Lagrange equation (1.8), which is given by

$$\frac{\mathrm{d}}{\mathrm{d}t}\frac{\partial\mathcal{L}}{\partial\dot{\theta}} = \frac{\partial\mathcal{L}}{\partial\theta} \; , \tag{6.2}$$

yields

$$\ddot{\theta} + \frac{g}{l}\sin\theta = 0 \; . \tag{6.3}$$

By using the approximation $x = l\sin\theta \simeq l\theta$, which is valid provided that $\theta \ll 1$, the equation of motion (6.3) becomes

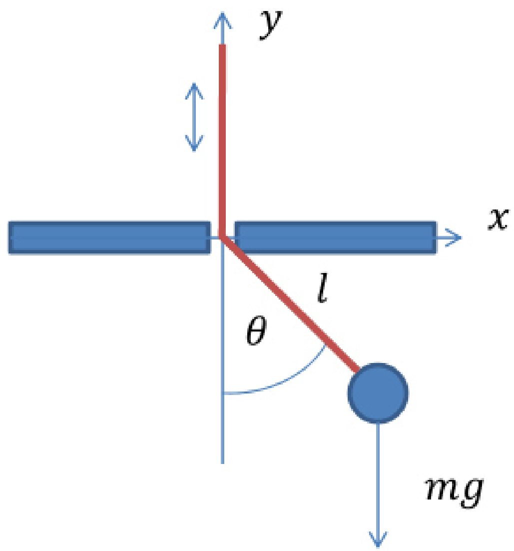


Fig. 6.1. Parametrically excited pendulum.

$$\ddot{x} + \omega^2 x = 0 \; , \tag{6.4}$$

where

$$\omega = \sqrt{\frac{g}{l}} \; . \tag{6.5}$$

Next we consider the case where ω is periodically modulated in time. This can be done by pulling up and down the clamping point that holds the string (see Fig. 6.1). Consider the case where by doing this the length of the pendulum l is forced to oscillate in time at angular frequency $2\omega_0$ according to

$$l(t) = l_0 \left(1 - \zeta \cos(2\omega_0 t)\right) \; , \tag{6.6}$$

where both l_0 and ζ are constants, and where

$$\omega_0 = \sqrt{\frac{g}{l_0}} \; . \tag{6.7}$$

As will be seen later, the angular frequency of the parametric modulation is chosen to be $2\omega_0$ in order to obtain a relatively large response. Assuming that $\zeta \ll 1$ one finds from Eq. (6.5) that the angular resonance frequency ω is consequently forced to oscillate in time according to [recall that $(1-\varepsilon)^{-1} = 1 + \varepsilon + O\left(\varepsilon^2\right)$]

$$\omega = \omega_0 \sqrt{1 + \zeta \cos(2\omega_0 t)} \ . \tag{6.8}$$

Exercise 6.1.2. Consider the case where the pendulum oscillates in time according to

$$x(t) = x_0 \cos(\omega_0 t + \phi) \ , \tag{6.9}$$

where both x_0 and ϕ are constants. Calculate the power P_{p} needed for moving the clamp.

Solution 6.1.2. The tension N in the string is given by $N = mg\cos\theta \simeq mg\left(1 - x^2/2l_0^2\right)$, thus the work W_{p} done by the moving the clamp per one period of oscillation is given by

$$\begin{aligned}
W_{\mathrm{p}} &= -\int_0^{2\pi/\omega_0} \mathrm{d}t\, \dot{l} N \\
&= -2\zeta m g \omega_0 l_0 \int_0^{2\pi/\omega_0} \mathrm{d}t\ \sin(2\omega_0 t) \left(1 - \frac{x_0^2 \cos^2(\omega_0 t + \phi)}{2l_0^2}\right) \\
&= \zeta m \omega_0^2 x_0^2 \int_0^{2\pi} \mathrm{d}\tau\ \sin(2\tau) \cos^2(\tau + \phi) \\
&= \frac{\zeta m \omega_0^2 x_0^2}{2} \int_0^{2\pi} \mathrm{d}\tau\ \sin(2\tau) \cos(2(\tau + \phi)) \\
&= -m\omega_0^2 x_0^2 \frac{\pi\zeta \sin(2\phi)}{2} \ .
\end{aligned} \tag{6.10}$$

The power P_{p} is found by dividing by the period time $2\pi/\omega_0$

$$P_{\mathrm{p}} = \frac{W_{\mathrm{p}}}{2\pi/\omega_0} = -m\omega_0^2 x_0^2 \gamma_{\mathrm{f}} \sin(2\phi) \ , \tag{6.11}$$

where

$$\gamma_{\mathrm{f}} = \frac{\omega_0 \zeta}{4} \ . \tag{6.12}$$

To account for damping the term $2\gamma\dot{x}$ is added to the equation of motion [see Eqs. (6.4) and (6.8)]

$$\ddot{x} + 2\gamma\dot{x} + \omega_0^2 (1 + \zeta \cos(2\omega_0 t))\, x = 0 \ . \tag{6.13}$$

Exercise 6.1.3. Calculate the power P_{d} that is dissipated from the oscillation.

Solution 6.1.3. The work W_{d} associated with the damping term per period is given by

$$
\begin{aligned}
W_{\mathrm{d}} &= -m \int_0^{2\pi/\omega_0} \mathrm{d}t\ \dot{x} \times (2\gamma \dot{x}) \\
&= -2m\omega_0^2 x_0^2 \gamma \int_0^{2\pi/\omega_0} \mathrm{d}t\ \sin^2(\omega_0 t + \phi) \\
&= -m\omega_0^2 x_0^2 \frac{2\pi\gamma}{\omega_0} \;,
\end{aligned}
\tag{6.14}
$$

thus the associated power P_{d} is given by

$$
P_{\mathrm{d}} = -m\omega_0^2 x_0^2 \gamma \;.
$$

Adding this result to P_{p} [see Eq. (6.11)] yields the total power added to the resonator

$$
P_{\mathrm{d}} + P_{\mathrm{p}} = -m\omega_0^2 x_0^2 (\gamma + \gamma_{\mathrm{f}} \sin(2\phi)) \;. \tag{6.15}
$$

Note that P_{p} can become positive provided that $\gamma_{\mathrm{f}} \geq \gamma$. For that case the power added to the system by the parametric excitation exceeds the power that is removed due to damping.

6.1.1 Equation of Motion

By adding an external forcing term $f(t)$ the equation of motion becomes

$$
\ddot{x} + 2\gamma\dot{x} + \omega_0^2 [1 + \zeta \cos(2\omega_0 t)] x = f(t) \;, \tag{6.16}
$$

or alternatively [see Eq. (3.35)]

$$
(\mathcal{D} - \Gamma)(\mathcal{D} - \Gamma^*) x + \omega_0^2 \zeta \cos(2\omega_0 t) x = f(t) \;, \tag{6.17}
$$

where $\mathcal{D} = \mathrm{d}/\mathrm{d}t$. As was done in the previous section, the variable

$$
a = (\mathcal{D} - \Gamma^*) x = \dot{x} - \Gamma^* x \tag{6.18}
$$

is introduced, where $\Gamma = -\gamma + i\sqrt{\omega_0^2 - \gamma^2}$ [see Eq. (3.7)]. By assuming the case of high quality factor (i.e. the case where $\gamma \ll \omega_0$) one finds that

$$
\Gamma \simeq -\gamma + i\omega_0 \;, \tag{6.19}
$$

and thus [see Eq. (3.45)]

$$
x = \frac{a - a^*}{\Gamma - \Gamma^*} \simeq \frac{a - a^*}{2i\omega_0} \;. \tag{6.20}
$$

With the help of Eq. (3.37) one finds that the equation of motion can be expressed as

$$\dot{a} - \Gamma a + \omega_0 \zeta \cos(2\omega_0 t) \frac{a - a^*}{2i} = f(t) \ . \tag{6.21}$$

The driving term $f(t)$ is assumed to be monochromatic. Consider the relatively simple case of driving at resonance, i.e. the case where $f(t)$ is assumed to be given by $Fe^{i\omega_0 t}$, where F is a complex constant. The transformation into the rotating frame is take to be given by

$$a = Ae^{i\omega_0 t} \ . \tag{6.22}$$

Substituting into Eq. (6.21) yields

$$\dot{A} + \gamma A - i\gamma_{\mathrm{f}} \left(e^{2i\omega_0 t} + e^{-2i\omega_0 t} \right) \left(A - A^* e^{-2i\omega_0 t} \right) = F \ , \tag{6.23}$$

where

$$\gamma_{\mathrm{f}} = \frac{\omega_0 \zeta}{4} \ . \tag{6.24}$$

6.1.2 Gain

The parametric term contains a constant term (given by $i\gamma_{\mathrm{f}} A^*$) and terms oscillating at angular frequencies $2\omega_0$ and $4\omega_0$. In the RWA the oscillating terms are disregarded, and the equation of motion is approximately taken to be given by

$$\dot{A} + \gamma A + i\gamma_{\mathrm{f}} A^* = F \ . \tag{6.25}$$

It is convenient to introduce the real variables A_1 and A_2

$$A_1 = Ae^{-\frac{i\pi}{4}} + A^* e^{\frac{i\pi}{4}} \ , \tag{6.26}$$

$$A_2 = Ae^{\frac{i\pi}{4}} + A^* e^{-\frac{i\pi}{4}} \ . \tag{6.27}$$

The inverse transformation is given by

$$A = \frac{A_1 e^{\frac{i\pi}{4}} + A_2 e^{-\frac{i\pi}{4}}}{2} \ , \tag{6.28}$$

$$A^* = \frac{A_1 e^{-\frac{i\pi}{4}} + A_2 e^{\frac{i\pi}{4}}}{2} \ . \tag{6.29}$$

By substituting A_1 and A_2 into Eq. (6.25) one finds two decoupled equations of motion

$$\dot{A}_1 + (\gamma + \gamma_{\mathrm{f}}) A_1 = F_1 \ , \tag{6.30}$$

$$\dot{A}_2 + (\gamma - \gamma_{\mathrm{f}}) A_2 = F_2 \ , \tag{6.31}$$

where the forcing terms F_1 and F_2 are given by

$$F_1 = Fe^{-\frac{i\pi}{4}} + F^* e^{\frac{i\pi}{4}} \ , \tag{6.32}$$

$$F_2 = Fe^{\frac{i\pi}{4}} + F^* e^{-\frac{i\pi}{4}} \ , \tag{6.33}$$

or by

$$F_1 = 2|F|\cos\left(\phi_{\mathrm{F}} - \frac{\pi}{4}\right) , \tag{6.34}$$

$$F_2 = 2|F|\cos\left(\phi_{\mathrm{F}} + \frac{\pi}{4}\right) , \tag{6.35}$$

where the following notation has been introduced

$$F = |F|\, e^{i\phi_{\mathrm{F}}} . \tag{6.36}$$

What is the physical meaning of the amplitudes A_1 and A_2? To answer this we express below the displacement x in terms of A_1 and A_2

$$\begin{aligned} x &= \frac{a - a^*}{2i\omega_0} \\ &= \frac{Ae^{i\omega_0 t} - A^* e^{-i\omega_0 t}}{2i\omega_0} \\ &= \frac{A_1 \sin\left(\omega_0 t + \frac{\pi}{4}\right) - A_2 \cos\left(\omega_0 t + \frac{\pi}{4}\right)}{2\omega_0} , \end{aligned} \tag{6.37}$$

thus A_1 and A_2 are amplitudes of two orthogonal quadratures. As can be seen from Eqs. (6.30) and (6.31), while the effective damping of the dynamics of A_1 is $\gamma + \gamma_{\mathrm{f}}$, the effective damping of the dynamics of A_2 is $\gamma - \gamma_{\mathrm{f}}$. We refer to A_1 as the deamplified quadrature and to A_2 as the amplified quadrature.

In steady state (i.e. when $\dot{A}_1 = 0$ and $\dot{A}_2 = 0$) A_1 and A_2 are given by [see Eqs. (6.30), (6.31), (6.34) and (6.35)]

$$A_1 = \frac{2|F|\cos\left(\phi_{\mathrm{F}} - \frac{\pi}{4}\right)}{\gamma + \gamma_{\mathrm{f}}} , \tag{6.38}$$

$$A_2 = \frac{2|F|\cos\left(\phi_{\mathrm{F}} + \frac{\pi}{4}\right)}{\gamma - \gamma_{\mathrm{f}}} . \tag{6.39}$$

The gain of the amplifier, which is defined by

$$G(\phi_{\mathrm{F}}) = \frac{\left(A_1^2 + A_2^2\right)_{\text{pump on}}^{1/2}}{\left(A_1^2 + A_2^2\right)_{\text{pump off}}^{1/2}} , \tag{6.40}$$

is thus given by

$$G(\phi_{\mathrm{F}}) = \sqrt{\frac{\cos^2\left(\phi_{\mathrm{F}} - \frac{\pi}{4}\right)}{\left(1 + \frac{\gamma_{\mathrm{f}}}{\gamma}\right)^2} + \frac{\cos^2\left(\phi_{\mathrm{F}} + \frac{\pi}{4}\right)}{\left(1 - \frac{\gamma_{\mathrm{f}}}{\gamma}\right)^2}} . \tag{6.41}$$

The gain G, which is plotted in Fig. 6.2 as a function of ϕ_{F} for different values of the ratio $\gamma_{\mathrm{f}}/\gamma$, is a periodic function of ϕ_{F}. The gain diverges when

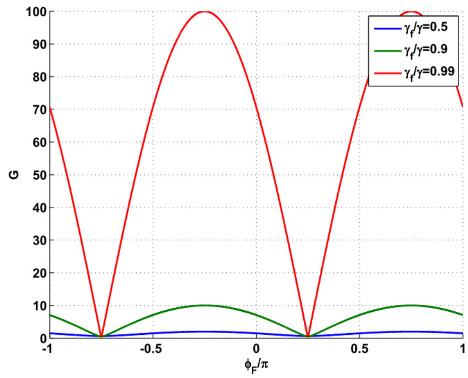


Fig. 6.2. The function $G(\phi_{\mathrm{F}})$.

the ratio $\gamma_{\mathrm{f}}/\gamma$ exceeds unity. For that case when $\gamma_{\mathrm{f}}/\gamma > 1$ the zero solution becomes unstable and the analysis above breaks down. In that region higher order nonlinear terms have to be taken into account in order to correctly determine the steady state solution.

6.2 Duffing Oscillator

Consider a mass m that can move along the x axis in one dimension. The mass is connected to a spring. The system constitutes a simple harmonic oscillator provided that the spring satisfies Hooke's law, i.e. provided that the restoring force is given by $-kx$, where k is the spring constant. The equation of motion is given by $\ddot{x} + \omega_0^2 x = 0$, where $\omega_0 = \sqrt{k/m}$. However, when the Hooke's law is violated the equation of motion may contain additional nonlinear terms. The exercise below demonstrates that even springs that satisfy Hooke's law can be used to construct a nonlinear oscillator.

Exercise 6.2.1. The mass that is seen in Fig. 6.3 is allowed to move along the x axis. Both springs that are seen in the figure are assumed to satisfy Hooke's law. The potential energy of each spring is given by $k(l - l_0)^2/2$, where l is the length of the spring and where l_0 is a constant. The distance between the two walls that clamp the springs is $2L$. Find an equation of motion for x.

Solution 6.2.1. The total potential energy is given by

$$U = k\left(\sqrt{L^2 + x^2} - l_0\right)^2 , \tag{6.42}$$

and therefore the Lagrangian is given by

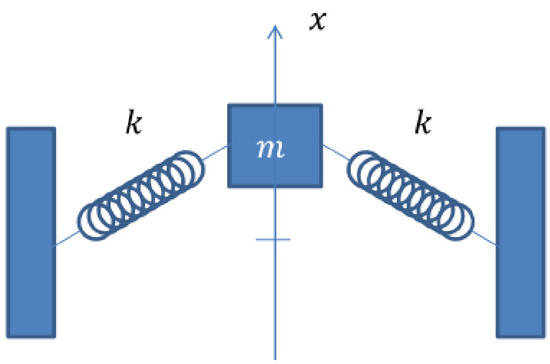


Fig. 6.3. Example of a Duffing oscillator.

$$\mathcal{L} = \frac{m\dot{x}^2}{2} - k\left(\sqrt{L^2 + x^2} - l_0\right)^2 . \tag{6.43}$$

The Euler-Lagrange equation (1.8), which is given by

$$\frac{\mathrm{d}}{\mathrm{d}t}\frac{\partial \mathcal{L}}{\partial \dot{x}} = \frac{\partial \mathcal{L}}{\partial x} , \tag{6.44}$$

yields the following equation of motion

$$\ddot{x} + \omega_0^2 x \frac{1 - \frac{l_0}{L\sqrt{1+\left(\frac{x}{L}\right)^2}}}{1 - \frac{l_0}{L}} = 0 , \tag{6.45}$$

where

$$\omega_0 = \sqrt{\frac{2k}{m}\left(1 - \frac{l_0}{L}\right)} . \tag{6.46}$$

For small amplitudes the system is expected to mimic a harmonic oscillator having angular resonance frequency ω_0. However, when the amplitude x becomes sufficiently large the nonlinearity of the restoring force is expected to play a role. To third order in x the equation of motion is given by

$$\ddot{x} + \omega_0^2\left(1 + \kappa x^2\right)x = 0 , \tag{6.47}$$

where

$$\kappa = \frac{l_0}{2L^3\left(1 - \frac{l_0}{L}\right)} . \tag{6.48}$$

6.2.1 Equation of Motion

By adding a damping term $2\gamma\dot{x}$ and a monochromatic forcing term $Fe^{i\omega_{\mathrm{p}}t}$ the equation of motion becomes

$$\ddot{x} + 2\gamma\dot{x} + \omega_0^2 \left(1 + \kappa x^2\right) x = Fe^{i\omega_{\mathrm{p}}t} . \tag{6.49}$$

In terms of the variable $a = (\mathcal{D} - \Gamma^*) x = \dot{x} - \Gamma^* x$ the equation of motion is given by [see Eqs. (3.36), (3.37) and (3.46)]

$$\dot{a} - \Gamma a + \omega_0^2 \kappa \left(\frac{a - a^*}{2i\omega_0}\right)^3 = Fe^{i\omega_{\mathrm{p}}t} . \tag{6.50}$$

The transformation into the rotating frame, which is performed by the substitution $a = Ae^{i\omega_{\mathrm{p}}t}$ [see Eq. (3.38)], yields [the approximation $\Gamma \simeq -\gamma + i\omega_0$ is employed, see Eq. (3.40)]

$$\dot{A} + \left[i\left(\omega_{\mathrm{p}} - \omega_0\right) + \gamma\right] A + \omega_0^2 \kappa e^{-i\omega_{\mathrm{p}}t} \left(\frac{Ae^{i\omega_{\mathrm{p}}t} - A^* e^{-i\omega_{\mathrm{p}}t}}{2i\omega_0}\right)^3 = F .$$

The nonlinear term (i.e. the term that is proportional to κ) contains a stationary term, and oscillating terms at angular frequencies $2\omega_{\mathrm{p}}$ and $4\omega_{\mathrm{p}}$. In the RWA the oscillating terms are disregarded and the equation of motion becomes

$$\dot{A} + \left[i\left(\omega_{\mathrm{p}} - \omega_0 - K\left|A\right|^2\right) + \gamma\right] A = F , \tag{6.51}$$

where

$$K = \frac{3\kappa}{8\omega_0} . \tag{6.52}$$

6.2.2 Steady States

In steady state (i.e. when $\dot{A} = 0$) one finds by taking the absolute value squared of both sides of Eq. (6.51) that the real variable $E = \left|A\right|^2$ satisfies the following equation

$$\left[\left(\omega_{\mathrm{p}} - \omega_0 - KE\right)^2 + \gamma^2\right] E = p . \tag{6.53}$$

where $p = \left|F\right|^2$. Finding E by solving the cubic polynomial Eq. (6.53) allows calculating A in steady state using Eq. (6.51)

$$A = \frac{F}{i\left(\omega_{\mathrm{p}} - \omega_0 - \frac{3\kappa E}{8\omega_0}\right) + \gamma} . \tag{6.54}$$

The cubic polynomial Eq. (6.53) for E can have either one, two or three different real roots, depending on the values of the detuning parameter $\omega_{\mathrm{p}} - \omega_0$ and the excitation amplitude p. Below we consider some special points of operation.

6.2.3 Special points

By taking the derivative of Eq. (6.53) with respect to the drive frequency ω_{p} one finds that

$$\frac{\partial E}{\partial \omega_{\mathrm{p}}} = -\frac{2E\left(\omega_{\mathrm{p}} - \omega_0 - KE\right)}{\left(\omega_{\mathrm{p}} - \omega_0 - 3KE\right)\left(\omega_{\mathrm{p}} - \omega_0 - KE\right) + \gamma^2} \ . \tag{6.55}$$

Similarly for the drive amplitude p

$$\frac{\partial E}{\partial p} = \frac{1}{\left(\omega_{\mathrm{p}} - \omega_0 - 3KE\right)\left(\omega_{\mathrm{p}} - \omega_0 - KE\right) + \gamma^2} \ . \tag{6.56}$$

The maximum of the function $E\left(\omega_{\mathrm{p}}\right)$. From Eq. (6.55) one finds that the maximum of the frequency response curve, i.e. the maximum of the function $E\left(\omega_{\mathrm{p}}\right)$ for a fixed drive amplitude p, occurs when

$$\omega_{\mathrm{p}} - \omega_0 - KE = 0 \ . \tag{6.57}$$

Thus the detuning $\omega_{\mathrm{p}} - \omega_0$ at which E obtains a maximum is shifted with respect to the linear case (i.e. the case where $K = 0$) from zero to $\omega_{\mathrm{p}} - \omega_0 = KE$. When $K > 0$ the system is said to exhibit 'hardening' behavior, whereas when $K < 0$ the system is said to exhibit 'softening' behavior.

The cusp point. At the cusp point the following holds

$$\frac{\partial \omega_{\mathrm{p}}}{\partial E} = \frac{\partial^2 \omega_{\mathrm{p}}}{\partial E^2} = 0 \ . \tag{6.58}$$

As we will seen later the onset of bistability occurs at that point. From Eq. (6.55) one finds that at the cusp point the following holds

$$\left(\omega_{\mathrm{p}} - \omega_0 - 3KE\right)\left(\omega_{\mathrm{p}} - \omega_0 - KE\right) + \gamma^2 = 0 \ , \tag{6.59}$$

and

$$\begin{aligned} 0 &= \frac{\partial}{\partial E}\left[\left(\omega_{\mathrm{p}} - \omega_0 - 3KE\right)\left(\omega_{\mathrm{p}} - \omega_0 - KE\right) + \gamma^2\right] \\ &= -4K\left(\omega_{\mathrm{p}} - \omega_0\right) + 6K^2 E \ . \end{aligned} \tag{6.60}$$

The values of E, of the detuning $\omega_{\mathrm{p}} - \omega_0$ and of the drive amplitude p at which these conditions are satisfied are labeled as E_{c}, $\left(\omega_{\mathrm{p}} - \omega_0\right)_{\mathrm{c}}$ and p_{c} respectively. From the last two Eqs. (6.59) and (6.60) one finds that [see also Eq. (6.53)]

$$E_{\mathrm{c}} = \frac{2\sqrt{3}\gamma}{3\left|K\right|} \ , \tag{6.61}$$

$$\left(\omega_{\mathrm{p}} - \omega_0\right)_{\mathrm{c}} = \sqrt{3}\gamma\frac{K}{\left|K\right|} \ , \tag{6.62}$$

$$p_{\mathrm{c}} = \frac{8\sqrt{3}\gamma^3}{9\left|K\right|} \ . \tag{6.63}$$

Figure (6.4) shows the frequency response curves $E^{1/2}$ vs. normalized detuning $(\omega_{\mathrm{p}}-\omega_0)/(\omega_{\mathrm{p}}-\omega_0)_{\mathrm{c}}$ for three different values of p, namely $p_{\mathrm{c}}/2$, p_{c} and $2p_{\mathrm{c}}$. The top panel shows the stability diagram of the system in the plane of the two control parameters, namely detuning $\omega_{\mathrm{p}}-\omega_0$ and drive amplitude p. The yellow region labels the region of bistability, where the response has two locally stable values. The cusp point occurs at $\omega_{\mathrm{p}}-\omega_0 = (\omega_{\mathrm{p}}-\omega_0)_{\mathrm{c}}$ and $p=p_{\mathrm{c}}$.

6.2.4 Basins of Attraction

In the bistable region Eq. (6.51) has 3 different steady state solutions, labeled as A_1, A_2 and A_3, where both stable solutions A_1 and A_3 are attractors, and the unstable solution A_2 is a saddle point. The bistable region in the plane of parameters (ω_{p}, p) is seen in the top panel of Fig. 6.4.

Figure 6.5 shows some flow lines that are obtained by integrating Eq. (6.51). The red and blue lines represent flow toward the attractors at A_1 and A_3 respectively. The green line is the seperatrix, namely the boundary between the basins of attraction of the attractors at A_1 and A_3. Figure 6.6 shows the flow map for the case where the stable solution A_1 is close to the saddle point A_2. The flow map demonstrates that in that region the flow becomes almost one dimensional.

6.3 Problems

1. **parametric excitation with detuning** - When a frequency detuning Δ_{p} in the parametric excitation and a frequency detuning $\Delta_{\mathrm{p}}+\Delta_{\mathrm{f}}$ in the forcing are taken into account Eq. (6.16) becomes

$$\ddot{x}+2\gamma\dot{x}+\omega_0^2\left[1+\zeta\cos\left(2\left(\omega_0+\Delta_{\mathrm{p}}\right)t\right)\right]x=f\left(t\right)\ , \tag{6.64}$$

where the forcing term $f(t)$ is given by

$$f\left(t\right)=Fe^{i\left(\omega_0+\Delta_{\mathrm{p}}+\Delta_{\mathrm{f}}\right)t}+F^{*}e^{-i\left(\omega_0+\Delta_{\mathrm{p}}+\Delta_{\mathrm{f}}\right)t}\ . \tag{6.65}$$

Assume that $\gamma \ll \omega_0$, $\Delta_{\mathrm{p}} \ll \omega_0$ and $\Delta_{\mathrm{f}} \ll \omega_0$.

a) Add the detuning terms to the equation of motion for the complex amplitude A (6.23).
b) Find a formal solution to the equation of motion for A.
c) Find the steady state solution for the equation for A.
d) Express the equation of motion for A as a set of two decoupled differential equations.
e) Express the equation of motion for A in cylindrical coordinates.

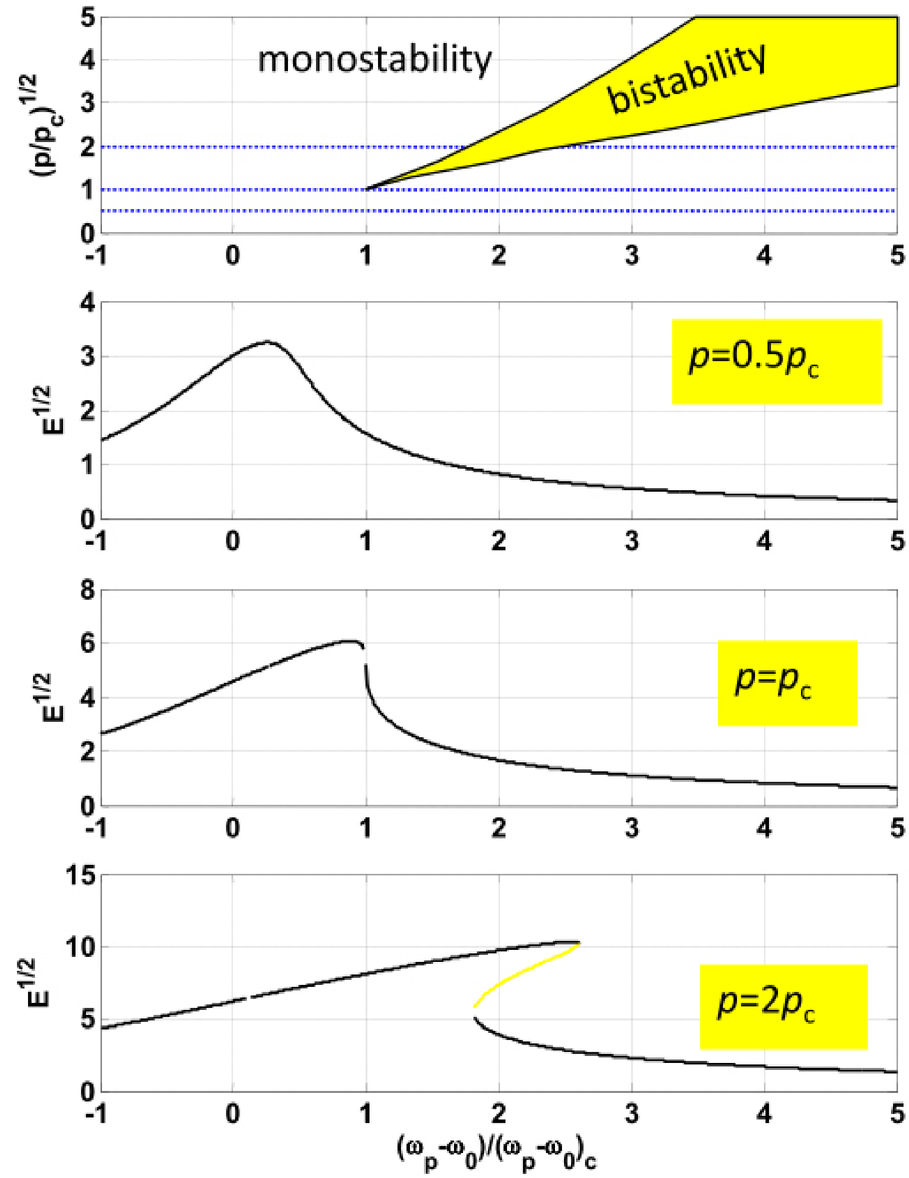


Fig. 6.4. Frequency response curves $E^{1/2}$ vs. normalized detuning $(\omega_{\mathrm{p}} - \omega_0) / (\omega_{\mathrm{p}} - \omega_0)_{\mathrm{c}}$ for three different values of p, namely $p_{\mathrm{c}}/2$, p_{c} and $2p_{\mathrm{c}}$. The top panel shows the stability diagram of the system in the plane of the two control parameters, namely detuning $\omega_{\mathrm{p}} - \omega_0$ and drive amplitude p.

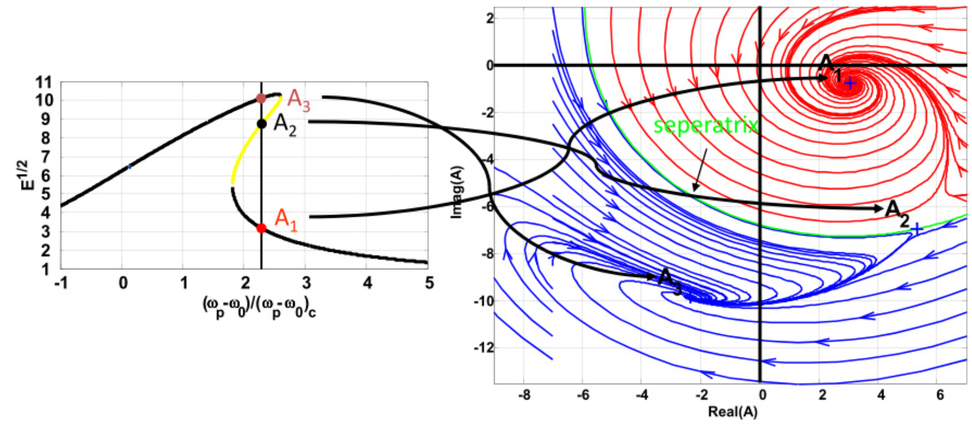


Fig. 6.5. Flow map in the complex plane of A obtained by integrating Eq. (6.51). The red and blue lines represent flow toward the attractors at A_1 and A_3 respectively. The green line is the seperatrix, namely the boundary between the basins of attraction of the attractors at A_1 and A_3.

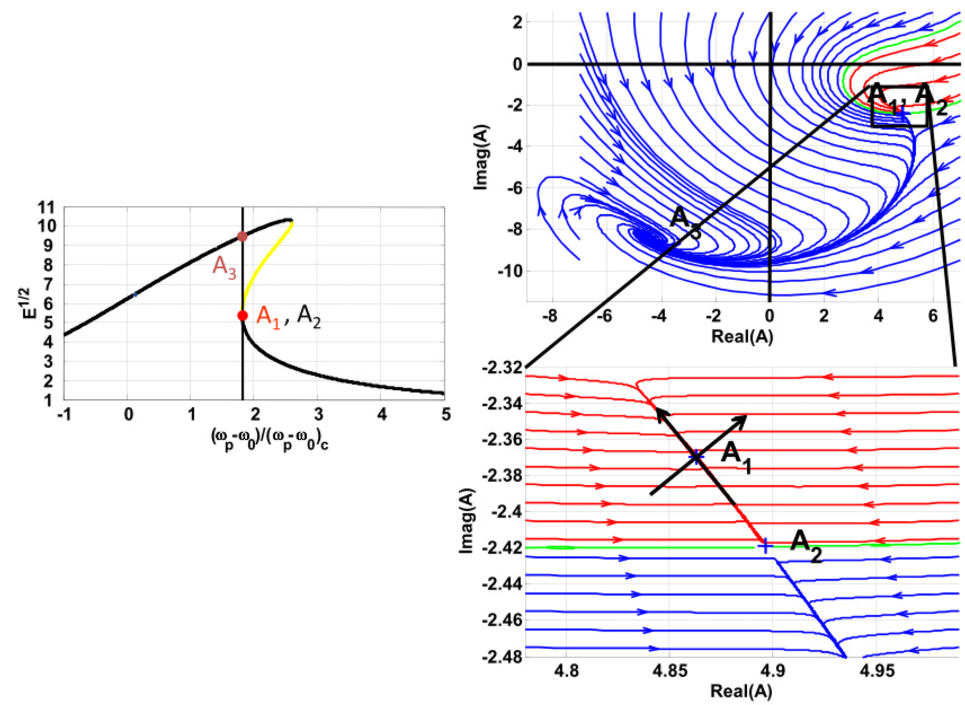


Fig. 6.6. Flow map for the case where the stable solution A_1 is close to the saddle point A_2.

2. To account for nonlinear mechanisms of damping an additional term is added to the evolution equation (6.51), which becomes

$$\dot{A}+\left[i\left(\omega_{\mathrm{p}}-\omega_0-K\left|A\right|^2\right)+\gamma+\gamma_3\left|A\right|^2\right]A=F\ , \tag{6.66}$$

where the coefficient of nonlinear damping γ_3 is assumed to be nonnegative, i.e. $\gamma_3\geq 0$. Generalize Eqs. (6.61), (6.62) and (6.63) for this case.

6.4 Solutions

1. The equation of motion (6.64) can be expressed as

$$(\mathcal{D}-\Gamma)\left(\mathcal{D}-\Gamma^*\right)x+\omega_0^2\zeta\cos\left(2\left(\omega_0+\Delta_{\mathrm{p}}\right)t\right)x=f\left(t\right)\ , \tag{6.67}$$

where $\mathcal{D}=\mathrm{d}/\mathrm{d}t$ and where $\Gamma=-\gamma+i\sqrt{\omega_0^2-\gamma^2}$. In terms of the complex variable a, which is given by

$$a=\left(\mathcal{D}-\Gamma^*\right)x=\dot{x}-\Gamma^*x\ , \tag{6.68}$$

one finds for the case where $\gamma\ll\omega_0$ (for which $\Gamma\simeq-\gamma+i\omega_0$) that

$$x=\frac{a-a^*}{\Gamma-\Gamma^*}\simeq\frac{a-a^*}{2i\omega_0}\ , \tag{6.69}$$

and thus [see Eq. (6.67)]

$$\dot{a}-\Gamma a+\omega_0\zeta\cos\left(2\left(\omega_0+\Delta_{\mathrm{p}}\right)t\right)\frac{a-a^*}{2i}=f\left(t\right)\ . \tag{6.70}$$

a) The transformation

$$a=Ae^{i(\omega_0+\Delta_{\mathrm{p}})t} \tag{6.71}$$

leads to

$$\begin{aligned}&\dot{A}+i\Delta_{\mathrm{p}}A+\gamma A\\&-i\gamma_{\mathrm{f}}\left(e^{2i(\omega_0+\Delta_{\mathrm{p}})t}+e^{-2i(\omega_0+\Delta_{\mathrm{p}})t}\right)\left(A-A^*e^{-2i(\omega_0+\Delta_{\mathrm{p}})t}\right)\\&=Fe^{i\Delta_{\mathrm{f}}t}+F^*e^{-i(2\omega_0+2\Delta_{\mathrm{p}}+\Delta_{\mathrm{f}})t}\ .\end{aligned} \tag{6.72}$$

where

$$\gamma_{\mathrm{f}}=\frac{\omega_0\zeta}{4}\ . \tag{6.73}$$

In the rotating wave approximation all the rapidly oscillating terms are disregarded

$$\dot{A}+i\Delta_{\mathrm{p}}A+\gamma A+i\gamma_{\mathrm{f}}A^*=Fe^{i\Delta_{\mathrm{f}}t}\ . \tag{6.74}$$

b) In a matrix form Eq. (6.74) can be expressed as

$$\frac{\mathrm{d}}{\mathrm{d}t}\begin{pmatrix} A \\ A^* \end{pmatrix} + M\begin{pmatrix} A \\ A^* \end{pmatrix} = \begin{pmatrix} Fe^{i\Delta_{\mathrm{f}}t} \\ F^*e^{-i\Delta_{\mathrm{f}}t} \end{pmatrix} , \tag{6.75}$$

where the matrix M is given by

$$M = \begin{pmatrix} i\Delta_{\mathrm{p}} + \gamma & i\gamma_{\mathrm{f}} \\ -i\gamma_{\mathrm{f}} & -i\Delta_{\mathrm{p}} + \gamma \end{pmatrix} . \tag{6.76}$$

The inverse matrix M^{-1} is given by

$$M^{-1} = \frac{1}{\gamma_{\mathrm{f}}^2 - \Delta_{\mathrm{p}}^2 - \gamma^2}\begin{pmatrix} i\Delta_{\mathrm{p}} - \gamma & i\gamma_{\mathrm{f}} \\ -i\gamma_{\mathrm{f}} & -i\Delta_{\mathrm{p}} - \gamma \end{pmatrix} . \tag{6.77}$$

The eigenvalues $m_\pm$ of M are given by

$$m_\pm = \gamma \pm \gamma_{\mathrm{f}}\sqrt{1 - \frac{\Delta_{\mathrm{p}}^2}{\gamma_{\mathrm{f}}^2}} , \tag{6.78}$$

and the following holds

$$D^{-1}MD = \begin{pmatrix} m_- & 0 \\ 0 & m_+ \end{pmatrix} , \tag{6.79}$$

where

$$D = \begin{pmatrix} -i\sqrt{1 - \frac{\Delta_{\mathrm{p}}^2}{\gamma_{\mathrm{f}}^2}} - \frac{\Delta_{\mathrm{p}}}{\gamma_{\mathrm{f}}} & i\sqrt{1 - \frac{\Delta_{\mathrm{p}}^2}{\gamma_{\mathrm{f}}^2}} - \frac{\Delta_{\mathrm{p}}}{\gamma_{\mathrm{f}}} \\ 1 & 1 \end{pmatrix} , \tag{6.80}$$

or

$$D = \begin{pmatrix} -ie^{-i\theta_\Delta} & ie^{i\theta_\Delta} \\ 1 & 1 \end{pmatrix} , \tag{6.81}$$

where

$$\theta_\Delta = \tan^{-1}\frac{\frac{\Delta_{\mathrm{p}}}{\gamma_{\mathrm{f}}}}{\sqrt{1 - \frac{\Delta_{\mathrm{p}}^2}{\gamma_{\mathrm{f}}^2}}} . \tag{6.82}$$

The solution of Eq. (6.75) can be expressed as

$$\begin{pmatrix} A(t) \\ A^*(t) \end{pmatrix} = \exp(-Mt)\begin{pmatrix} A(0) \\ A^*(0) \end{pmatrix} + \int_0^t \mathrm{d}t'\ \exp(M(t'-t))\begin{pmatrix} Fe^{i\Delta_{\mathrm{f}}t'} \\ F^*e^{-i\Delta_{\mathrm{f}}t'} \end{pmatrix} . \tag{6.83}$$

The matrix M can be expressed as

$$M = \gamma \mathbf{1} + \phi_{\mathrm{u}} M_{\mathrm{u}} \ , \tag{6.84}$$

where $\mathbf{1}$ is the 2×2 unity matrix, $\phi_{\mathrm{u}} = \left(\gamma_{\mathrm{f}}^2 - \Delta_{\mathrm{p}}^2\right)^{1/2}$ and where the matrix M_{u}, which is given by

$$M_{\mathrm{u}} = \phi_{\mathrm{u}}^{-1} \begin{pmatrix} i\Delta_{\mathrm{p}} & i\gamma_{\mathrm{f}} \\ -i\gamma_{\mathrm{f}} & -i\Delta_{\mathrm{p}} \end{pmatrix} \ , \tag{6.85}$$

satisfies the relation $M_{\mathrm{u}}^2 = \mathbf{1}$. Therefore

$$\begin{aligned} \exp(Mt) &= \exp(\gamma t) \sum_{n=0}^{\infty} \frac{(\phi_{\mathrm{u}} t M_{\mathrm{u}})^n}{n!} \\ &= \exp(\gamma t) \left[\mathbf{1} \cos(\phi_{\mathrm{u}} t) + M_{\mathrm{u}} \sin(\phi_{\mathrm{u}} t)\right] \ . \end{aligned} \tag{6.86}$$

c) Expressing A as

$$A = A_1 e^{-i\Delta_{\mathrm{f}} t} + A_2 e^{i\Delta_{\mathrm{f}} t} \ , \tag{6.87}$$

and substituting into Eq. (6.74) yields

$$\dot{A}_1 - i\Delta_{\mathrm{f}} A_1 + i\Delta_{\mathrm{p}} A_1 + \gamma A_1 + i\gamma_{\mathrm{f}} A_2^* = F \ , \tag{6.88}$$

$$\dot{A}_2 + i\Delta_{\mathrm{f}} A_2 + i\Delta_{\mathrm{p}} A_2 + \gamma A_2 + i\gamma_{\mathrm{f}} A_1^* = 0 \ . \tag{6.89}$$

The steady state solution is given by

$$A_1 = \frac{F}{-i\Delta_{\mathrm{f}} + i\Delta_{\mathrm{p}} + \gamma - \frac{\gamma_{\mathrm{f}}^2}{-i\Delta_{\mathrm{f}} - i\Delta_{\mathrm{p}} + \gamma}} \ , \tag{6.90}$$

$$A_2 = \frac{-i\gamma_{\mathrm{f}} A_1^*}{i\Delta_{\mathrm{f}} + i\Delta_{\mathrm{p}} + \gamma} \ . \tag{6.91}$$

d) In terms of the variables B_1 and B_2, which are defined by

$$\begin{pmatrix} B_1 \\ B_2 \end{pmatrix} = \begin{pmatrix} e^{-i\frac{\pi + 2\theta_\Delta}{4}} & 0 \\ 0 & e^{i\frac{\pi + 2\theta_\Delta}{4}} \end{pmatrix} D^{-1} \begin{pmatrix} A \\ A^* \end{pmatrix} \ , \tag{6.92}$$

Eq. (6.75) can be rewritten as [see Eq. (6.79)]

$$\begin{pmatrix} \frac{\mathrm{d}B_1}{\mathrm{d}t} + m_- B_1 \\ \frac{\mathrm{d}B_2}{\mathrm{d}t} + m_+ B_2 \end{pmatrix} = \begin{pmatrix} e^{-i\frac{\pi + 2\theta_\Delta}{4}} & 0 \\ 0 & e^{i\frac{\pi + 2\theta_\Delta}{4}} \end{pmatrix} D^{-1} \begin{pmatrix} F e^{i\Delta_{\mathrm{f}} t} \\ F^* e^{-i\Delta_{\mathrm{f}} t} \end{pmatrix} \ . \tag{6.93}$$

The following holds [see Eq. (6.81)]

$$D^{-1} = \frac{1}{2\cos\theta_{\Delta}} \begin{pmatrix} i & e^{i\theta_{\Delta}} \\ -i & e^{-i\theta_{\Delta}} \end{pmatrix} , \tag{6.94}$$

thus

$$\begin{pmatrix} e^{-i\frac{\pi+2\theta_{\Delta}}{4}} & 0 \\ 0 & e^{i\frac{\pi+2\theta_{\Delta}}{4}} \end{pmatrix} D^{-1} = \frac{\begin{pmatrix} e^{i\frac{\pi-2\theta_{\Delta}}{4}} & e^{-i\frac{\pi-2\theta_{\Delta}}{4}} \\ e^{-i\frac{\pi-2\theta_{\Delta}}{4}} & e^{i\frac{\pi-2\theta_{\Delta}}{4}} \end{pmatrix}}{2\cos\theta_{\Delta}} , \tag{6.95}$$

and therefore

$$\frac{\mathrm{d}B_1}{\mathrm{d}t} + m_- B_1 = \frac{e^{i\frac{\pi-2\theta_{\Delta}}{4}} F e^{i\Delta_{\mathrm{f}} t} + e^{-i\frac{\pi-2\theta_{\Delta}}{4}} F^* e^{-i\Delta_{\mathrm{f}} t}}{2\cos\theta_{\Delta}} , \tag{6.96}$$

$$\frac{\mathrm{d}B_2}{\mathrm{d}t} + m_+ B_2 = \frac{e^{-i\frac{\pi-2\theta_{\Delta}}{4}} F e^{i\Delta_{\mathrm{f}} t} + e^{i\frac{\pi-2\theta_{\Delta}}{4}} F^* e^{-i\Delta_{\mathrm{f}} t}}{2\cos\theta_{\Delta}} . \tag{6.97}$$

e) The complex amplitude $A = A_x + iA_y$, where A_x and A_y are both real, is expressed in cylindrical coordinates as $A = A_r e^{iA_\theta}$, where $A_r = \sqrt{A_x^2 + A_y^2}$ is positive and $A_\theta = \tan^{-1}(A_y/A_x)$ is real. With the help of the relations

$$\dot{A}_r = \frac{\partial A_r}{\partial A_x}\dot{A}_x + \frac{\partial A_r}{\partial A_y}\dot{A}_y = \frac{A_x\dot{A}_x + A_y\dot{A}_y}{A_r} , \tag{6.98}$$

and

$$\dot{A}_\theta = \frac{\partial A_\theta}{\partial A_x}\dot{A}_x + \frac{\partial A_\theta}{\partial A_y}\dot{A}_y = \frac{-A_y\dot{A}_x + A_x\dot{A}_y}{A_r^2} , \tag{6.99}$$

and the notation

$$F = F_r e^{iF_\theta} , \tag{6.100}$$

where F_r is positive and F_θ is real, one finds that [see Eq. (6.74)]

$$\begin{aligned} \dot{A}_r &= -\left(\gamma + \gamma_{\mathrm{f}} \sin\left(2A_\theta\right)\right) A_r \\ &\quad + F_r \cos\left(F_\theta + \Delta_{\mathrm{f}} t - A_\theta\right) , \end{aligned} \tag{6.101}$$

and

$$\begin{aligned} \dot{A}_\theta &= -\Delta_{\mathrm{p}} - \gamma_{\mathrm{f}} \cos\left(2A_\theta\right) \\ &\quad + \frac{F_r \sin\left(F_\theta + \Delta_{\mathrm{f}} t - A_\theta\right)}{A_r} . \end{aligned} \tag{6.102}$$

The transformation

$$A_\theta = A_\phi + \Delta_{\mathrm{f}} t \;, \tag{6.103}$$

leads to

$$\begin{aligned}\dot{A}_r &= -\left(\gamma + \gamma_{\mathrm{f}} \sin\left(2\left(A_\phi + \Delta_{\mathrm{f}} t\right)\right)\right) A_r \\ &\quad + F_r \cos\left(F_\theta - A_\phi\right) \;,\end{aligned} \tag{6.104}$$

and

$$\begin{aligned}\dot{A}_\phi &= -\Delta_{\mathrm{p}} - \Delta_{\mathrm{f}} - \gamma_{\mathrm{f}} \cos\left(2\left(A_\phi + \Delta_{\mathrm{f}} t\right)\right) \\ &\quad + \frac{F_r \sin\left(F_\theta - A_\phi\right)}{A_r} \;.\end{aligned} \tag{6.105}$$

Note that for the case where

$$\Delta_{\mathrm{p}} = \Delta_{\mathrm{f}} = \frac{\Delta_0}{2} \;, \tag{6.106}$$

Eqs. (6.104) and (6.105) become

$$\begin{aligned}&\frac{\dot{A}_r}{A_r} + \gamma + \gamma_{\mathrm{f}} \sin\left(2A_\phi + \Delta_0 t\right) \\ &= \frac{F_r \cos\left(F_\theta - A_\phi\right)}{A_r} \;,\end{aligned} \tag{6.107}$$

and

$$\begin{aligned}&\dot{A}_\phi + \Delta_0 + \gamma_{\mathrm{f}} \cos\left(2A_\phi + \Delta_0 t\right) \\ &= \frac{F_r \sin\left(F_\theta - A_\phi\right)}{A_r} \;.\end{aligned} \tag{6.108}$$

In steady state, i.e. when $\dot{A}_r = 0$ and $\dot{A}_\phi = 0$, these equations can be rewritten as

$$\frac{\frac{\Delta_0}{\gamma} + \frac{\gamma_{\mathrm{f}}}{\gamma} \cos\left(2A_\phi + \Delta_0 t\right)}{1 + \frac{\gamma_{\mathrm{f}}}{\gamma} \sin\left(2A_\phi + \Delta_0 t\right)} = \tan\left(F_\theta - A_\phi\right) \;, \tag{6.109}$$

and

$$A_r^2 = \frac{\gamma^{-2} F_r^2}{\left(1 + \frac{\gamma_{\mathrm{f}} \sin(2A_\phi + \Delta_0 t)}{\gamma}\right)^2 + \left(\frac{\Delta_0}{\gamma} + \frac{\gamma_{\mathrm{f}} \cos(2A_\phi + \Delta_0 t)}{\gamma}\right)^2} \;. \tag{6.110}$$

For the case $\gamma_{\mathrm{f}} = 0$ the steady state solution is given by

$$A_r = \frac{F_r}{\sqrt{\gamma^2 + \varDelta_0^2}} \,, \tag{6.111}$$

$$A_\phi = F_\theta - \tan^{-1}\frac{\varDelta_0}{\gamma} \,, \tag{6.112}$$

whereas for the case where $\varDelta_0 = 0$ and $F_\theta = 0$ the steady state solution is given by

$$A_r = \frac{\sqrt{1 + \frac{\gamma_{\mathrm{f}}^2}{\gamma^2}}}{\gamma\left(1 - \frac{\gamma_{\mathrm{f}}^2}{\gamma^2}\right)} F_r \,, \tag{6.113}$$

$$A_\phi = -\tan^{-1}\frac{\gamma_{\mathrm{f}}}{\gamma} \,. \tag{6.114}$$

For the later case instability occurs when $\gamma_{\mathrm{f}} \geq \gamma$.

2. In steady state Eq. (6.66) yields

$$\left[(\omega_{\mathrm{p}} - \omega_0 - KE)^2 + (\gamma + \gamma_3 E)^2\right] E = p \,, \tag{6.115}$$

where $E = |A|^2$ and where $p = |F|^2$. At the cusp point the following holds

$$\frac{\partial \omega_{\mathrm{p}}}{\partial E} = \frac{\partial^2 \omega_{\mathrm{p}}}{\partial E^2} = 0 \,. \tag{6.116}$$

The first requirement that $\partial\omega_{\mathrm{p}}/\partial E = 0$ leads to

$$0 = -\frac{(\omega_{\mathrm{p}} - \omega_0 - 3KE)(\omega_{\mathrm{p}} - \omega_0 - KE) + (\gamma + 3\gamma_3 E)(\gamma + \gamma_3 E)}{2(\omega_{\mathrm{p}} - \omega_0 - KE) E} \,, \tag{6.117}$$

or

$$0 = (\omega_{\mathrm{p}} - \omega_0 - 3KE)(\omega_{\mathrm{p}} - \omega_0 - KE) + (\gamma + 3\gamma_3 E)(\gamma + \gamma_3 E) \,, \tag{6.118}$$

whereas the second requirement $\partial^2\omega_{\mathrm{p}}/\partial E^2 = 0$ leads to

$$\begin{aligned} 0 &= \frac{\partial}{\partial E}\left[(\omega_{\mathrm{p}} - \omega_0 - 3KE)(\omega_{\mathrm{p}} - \omega_0 - KE) + (\gamma + 3\gamma_3 E)(\gamma + \gamma_3 E)\right] \\ &= -4K(\omega_{\mathrm{p}} - \omega_0) + 6\left(K^2 + \gamma_3^2\right) E + 4\gamma\gamma_3 \,. \end{aligned} \tag{6.119}$$

The values of E, of the detuning $\omega_{\mathrm{p}} - \omega_0$ and of the drive amplitude p at which these conditions are satisfied are labeled as E_{c}, $(\omega_{\mathrm{p}} - \omega_0)_{\mathrm{c}}$ and p_{c} respectively. From the last two equations one finds that

$$E_{\mathrm{c}} = \frac{2\sqrt{3}\gamma}{3K} \frac{\pm 1 + \frac{\sqrt{3}\gamma_3}{K}}{1 - \left(\frac{\sqrt{3}\gamma_3}{K}\right)^2} \ . \tag{6.120}$$

Recall that by definition $E = |A|^2$, i.e. $E \geq 0$, and that $\gamma_3 \geq 0$. On the other hand K can be either positive or negative. Hence, for $\sqrt{3}\gamma_3 < |K|$ the nonnegative solution for E_{c} is given by

$$E_{\mathrm{c}} = \frac{2\sqrt{3}\gamma}{3|K|} \frac{1}{1 - \frac{\sqrt{3}\gamma_3}{|K|}} \ . \tag{6.121}$$

For the other case for which $\sqrt{3}\gamma_3 > |K|$ no physical solution exists since both solutions are negative. We thus conclude that bistability is possible only when $\sqrt{3}\gamma_3 < |K|$. Using Eq. (6.121) one finds that

$$(\omega_{\mathrm{p}} - \omega_0)_{\mathrm{c}} = \sqrt{3}\gamma \frac{K}{|K|} \frac{1 + \frac{\gamma_3}{\sqrt{3}|K|}}{1 - \frac{\sqrt{3}\gamma_3}{|K|}} \ , \tag{6.122}$$

and [see Eq. (6.115)]

$$p_{\mathrm{c}} = \frac{8\sqrt{3}\gamma^3}{9|K|} \frac{1 + \frac{\gamma_3^2}{K^2}}{\left(1 - \frac{\sqrt{3}\gamma_3}{|K|}\right)^3} \ . \tag{6.123}$$

7. Elasticity

In this chapter we discuss the deformation of substances due to externally applied forces and/or due to change in their temperature ΔT (with respect to a reference temperature). Below we will define the normalized applied forces as components of *stress*, the normalized temperature change as thermal stress, and the resultant deformation as components of *strain*. We will make the following assumptions:

- The material is assumed to be both *uniform* and *isotropic*.
- The material is assumed to be *elastic*, namely the stress-strain response is assumed to be reversible.
- The stress-strain relations are assumed to be *linear*, i.e. it is assume that Hooke's law holds.
- The relative deformation (i.e. the strain) is assumed to be small.

7.1 Normal Stress

Consider the rectangular block that is seen in Fig. 7.1. A force F is applied normal to the right and to the left faces. When the force is pointing outwards it is said to be *tensile* (like in Fig. 7.1), whereas when the force is pointing inwards it is said to be *compressive*. The tensile force results in elongation in the longitudinal direction $L_l \rightarrow L_l + \Delta L_l$ (see Fig. 7.1). The assumption that the deformation is linear implies that ΔL_l is proportional to F. For a given F the change ΔL_l is expected to be proportional to L_l. Moreover, for a given ΔL_l the force F is expected to be proportional to the area A of the face. Therefore, the stress σ, which is defined to be the force per unit area $\sigma = F/A$, is expected to be proportional to the strain ϵ, which is defined to be the relative elongation $\epsilon = \Delta L_l / L_l$, i.e.

$$\underbrace{\frac{F}{A}}_{\text{stress, }\sigma} = E \underbrace{\frac{\Delta L_l}{L_l}}_{\text{strain, }\epsilon} , \tag{7.1}$$

and the proportionality factor, which is called the *Young's modulus* E, is expected to be a property of the material only (and not of the dimensions of the block).

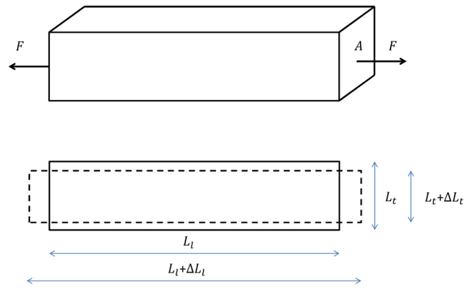


Fig. 7.1. Normal stress applied to rectangular block.

Due to the elongation in the direction of the applied normal stress, the block is expected to get skinnier in the two perpendicular directions. The ratio between the relative transverse contraction $\Delta L_{\mathrm{t}}/L_{\mathrm{t}}$ and the relative longitudinal stretching $\Delta L_{\mathrm{l}}/L_{\mathrm{l}}$ is expected to be a material specific constant (independent on the dimensions of the block). The absolute value of this ratio (note that it is expected that $\Delta L_{\mathrm{l}}\Delta L_{\mathrm{t}} \leq 0$) is called the Poisson's ratio ν, i.e.

$$\frac{\Delta L_{\mathrm{t}}}{L_{\mathrm{t}}} = -\nu \frac{\Delta L_{\mathrm{l}}}{L_{\mathrm{l}}} \ . \tag{7.2}$$

The Cartesian components of normal stress are denoted by σ_{xx}, σ_{yy} and σ_{zz} and the Cartesian components of normal strain are denoted by ϵ_{xx}, ϵ_{yy} and ϵ_{zz}. The stress-strain relations (7.1) and (7.2) between normal stress and normal strain can be written in a matrix form as

$$\begin{pmatrix} \epsilon_{xx} \\ \epsilon_{yy} \\ \epsilon_{zz} \end{pmatrix} = \frac{1}{E} \begin{pmatrix} 1 & -\nu & -\nu \\ -\nu & 1 & -\nu \\ -\nu & -\nu & 1 \end{pmatrix} \begin{pmatrix} \sigma_{xx} \\ \sigma_{yy} \\ \sigma_{zz} \end{pmatrix} \ . \tag{7.3}$$

7.2 The Bulk Modulus

When normal stress is applied and when $\sigma_{xx} = \sigma_{yy} = \sigma_{zz}$ the stress F/A is said to hydrostatic. The bulk modulus K is defined as the ratio between hydrostatic stress F/A and relative volume compression $\Delta V/V$

$$K = \frac{F/A}{\Delta V/V} \ . \tag{7.4}$$

Exercise 7.2.1. Express the bulk modulus K in terms of the Young's modulus E and the Poisson's ratio ν.

Solution 7.2.1. With the help of Eq. (7.3) one finds for the case of hydrostatic stress that $\epsilon_{xx} = \epsilon_{yy} = \epsilon_{zz} = \sigma (1 - 2\nu) / E$, where $\sigma = \sigma_{xx} = \sigma_{yy} = \sigma_{zz} = F/A$. Recall that the strain is assumed to be small, thus the relative change in volume $\Delta V/V = (1 + \epsilon_{xx}) (1 + \epsilon_{yy}) (1 + \epsilon_{zz}) - 1$ is approximately given by $\Delta V/V \simeq \epsilon_{xx} + \epsilon_{yy} + \epsilon_{zz}$, and consequently the bulk modulus K is given by

$$K = \frac{E}{3(1-2\nu)} \; . \tag{7.5}$$

7.3 The Shear Modulus

Contrary to the case of normal strain, where the external forces are applied normally to faces, in the case of shear strain the forces are applied parallelly. Consider the cube seen in Fig. 7.2. A force F_{t} is applied to the top face. The force is parallel to the top pace, however it is normal to the side faces on the left and on the right. The dotted line in Fig. 7.2 sketches the twisted shape of the cube due to the applied shear strain. The ratio δ / L_{t} (Fig. 7.2), which for the case of small strain represents the angle of twisting of the side face, is expected to be proportional to F_{t} (due to the assumption of linear response). Moreover, it is easy to see that for a given ratio δ / L_{t} the force F_{t} is expected to be proportional to the area A_{t} of the top side of the cube. Therefore, the shear stress τ, which is defined to be the force per unit area $\tau = F_{\mathrm{t}} / A_{\mathrm{t}}$, is expected to be proportional to the shear strain γ, which is defined to be the twisting angle $\gamma = \delta / L_{\mathrm{t}}$, i.e.

$$\underbrace{\frac{F_{\mathrm{t}}}{A_{\mathrm{t}}}}_{\text{shear stress, } \tau} = G \underbrace{\frac{\delta}{L_{\mathrm{t}}}}_{\text{shear strain, } \gamma} , \tag{7.6}$$

and the proportionality factor, which is called the *shear modulus* G, is expected to be a property of the material only (and not of the dimensions of the block).

Exercise 7.3.1. Express the shear modulus G in terms of the Young's modulus E and the Poisson's ratio ν.

Solution 7.3.1. Consider a cube having edges of length L_{s} [see Fig. 7.3(a)]. A shear stress is applied to the top, bottom, left and right sides. Note that the force magnitude (labelled as F_{t}) is assumed to be the same for all four faces to ensure that the total force and the total torque vanish. In the absent of the shear stress that is applied to the left and right sides the situation is equivalent to the one seen in Fig. (7.2), where the cube is attached to a 'table' beneath it, which effectively applies a shear stress to the bottom side. However, as will be shown below, by adding the shear stress to the left and

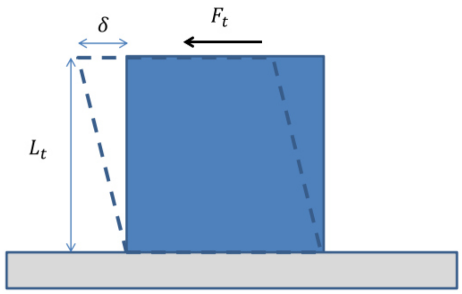


Fig. 7.2. A cube under shear strain.

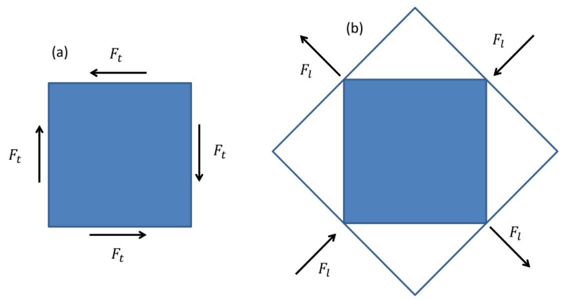


Fig. 7.3. Equivalency between shear and normal stress.

right sides [see Fig. 7.3(a)] the total shear stress can be described in terms of a combination of tensile and compressive normal stresses. To demonstrate this point, consider the case where the cube is embedded in a larger rectangular block, as seen in Fig. 7.3(b). The side in the plane of Fig. 7.3(b) of the rectangular block has a square shape (rotated by $\pi/4$ with respect to the face of the smaller cube in the same plane) having edge length of $L_{\mathrm{b}} = \sqrt{2}L_{\mathrm{s}}$. Applying normal stress to the sides of the rectangular block in the way that is indicated in Fig. 7.3(b) results in effective shear stress acting on the smaller cube. This can be seen by considering the total force acting on each of the 4 triangular prisms that are seen in Fig. 7.3(b). This consideration also leads to the conclusion that the effective shear stress in both cases can be made identical provided that the longitudinal force F_{l} that is normally applied to the sides of the big cube is taken to be given by

$$F_{\mathrm{l}} = \sqrt{2}F_{\mathrm{t}} \; . \tag{7.7}$$

Thus the corresponding normal stress that is applied to the sides of the big cube is given by

$$\sigma = \frac{F_{\mathrm{l}}}{L_{\mathrm{b}}L_{\mathrm{s}}} = \tau_{\mathrm{s}} \; , \tag{7.8}$$

where $\tau_{\mathrm{s}} = F_{\mathrm{t}}/L_{\mathrm{s}}^2$ is the shear stress that is applied to the small cube. The normal stress is tensile in one direction, which will be taken to be the x direction, and it is compressing in the perpendicular direction, which is taken to be the y direction. The resultant strain can be evaluated using Eq. (7.3)

$$\begin{pmatrix} \epsilon_{xx} \\ \epsilon_{yy} \\ \epsilon_{zz} \end{pmatrix} = \frac{1}{E} \begin{pmatrix} 1 & -\nu & -\nu \\ -\nu & 1 & -\nu \\ -\nu & -\nu & 1 \end{pmatrix} \begin{pmatrix} \sigma \\ -\sigma \\ 0 \end{pmatrix} = \begin{pmatrix} \frac{\sigma(1+\nu)}{E} \\ -\frac{\sigma(1+\nu)}{E} \\ 0 \end{pmatrix} . \tag{7.9}$$

The strain deforms the square shape of the sides of the small cube in the xy plane into a rhombus having two diagonals of length $D \pm \Delta D$, where $D = \sqrt{2} L_{\mathrm{s}}$ and where

$$\frac{\Delta D}{D} = \frac{\sigma(1+\nu)}{E} = \frac{\tau_{\mathrm{s}}(1+\nu)}{E} . \tag{7.10}$$

On the other hand, it is easy to see from Fig. 7.2 that for the case of small strain, i.e. for the case where $\Delta D \ll D$, the twisting angle of the square side (which is labelled by δ/L_{t} in Fig. 7.2, where in the present case L_{t} represents the edge length of the small cube L_{s}) is related to ΔD by

$$\frac{\delta}{L_{\mathrm{s}}} = \frac{\sqrt{2}\Delta D}{L_{\mathrm{s}}} , \tag{7.11}$$

thus with the help of Eq. (7.10) and the relation $D = \sqrt{2} L_{\mathrm{s}}$ one finds that

$$\frac{\delta}{L_{\mathrm{s}}} = \frac{2\Delta D}{D} = \frac{2\tau_{\mathrm{s}}(1+\nu)}{E} , \tag{7.12}$$

and therefore the shear modulus [see Eq. (7.6)] is given by

$$G = \frac{E}{2(1+\nu)} . \tag{7.13}$$

7.4 Thermal Stress

Consider a rectangular block. At a reference temperature T_0 the length of the block in one direction is L_0. In the absent of any externally applied forces a temperature change $\Delta T = T - T_0$ with respect to the reference temperature results in a length change $\Delta L = L - L_0$. The corresponding thermal strain ϵ_{T} is

$$\epsilon_{\mathrm{T}} = \alpha \Delta T , \tag{7.14}$$

where α is the material-specific coefficient of thermal expansion. In the presence of stress the thermal strain ϵ_{T} should be added to the stress-induced strain. Adding thermal strain terms to Eq. (7.3) yields

$$\begin{pmatrix} \epsilon_{xx} \\ \epsilon_{yy} \\ \epsilon_{zz} \end{pmatrix} = \frac{1}{E} \begin{pmatrix} 1 & -\nu & -\nu \\ -\nu & 1 & -\nu \\ -\nu & -\nu & 1 \end{pmatrix} \begin{pmatrix} \sigma_{xx} \\ \sigma_{yy} \\ \sigma_{zz} \end{pmatrix} + \alpha \Delta T \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix} . \tag{7.15}$$

7.5 Problems

1. Calculate the ratio $\sigma_{xx}/\epsilon_{xx}$ for the case where the lateral strain is constrained to vanish, i.e. $\epsilon_{yy} = \epsilon_{zz} = 0$.

7.6 Solutions

1. By inverting Eq. (7.3) one finds in general that

$$\begin{pmatrix} \sigma_{xx} \\ \sigma_{yy} \\ \sigma_{zz} \end{pmatrix} = \frac{E\left(1-\nu\right)}{\left(1+\nu\right)\left(1-2\nu\right)} \begin{pmatrix} 1 & \frac{\nu}{1-\nu} & \frac{\nu}{1-\nu} \\ \frac{\nu}{1-\nu} & 1 & \frac{\nu}{1-\nu} \\ \frac{\nu}{1-\nu} & \frac{\nu}{1-\nu} & 1 \end{pmatrix} \begin{pmatrix} \epsilon_{xx} \\ \epsilon_{yy} \\ \epsilon_{zz} \end{pmatrix} , \tag{7.16}$$

thus for the present case where $\epsilon_{yy} = \epsilon_{zz} = 0$ the following holds

$$\begin{pmatrix} \frac{\sigma_{xx}}{\epsilon_{xx}} \\ \frac{\sigma_{yy}}{\epsilon_{xx}} \\ \frac{\sigma_{zz}}{\epsilon_{xx}} \end{pmatrix} = \frac{E\left(1-\nu\right)}{\left(1+\nu\right)\left(1-2\nu\right)} \begin{pmatrix} 1 \\ \frac{\nu}{1-\nu} \\ \frac{\nu}{1-\nu} \end{pmatrix} . \tag{7.17}$$

8. Beams and Strings

This chapter is devoted to the mechanical properties of beams. Elastic instability and buckling are studied. The limit of a string is obtained when the effect of stiffness can be disregarded.

8.1 Bending

Consider a small section of a beam that is bent in the xy plane, as seen in Fig. 8.1. The curvature κ due to the bending is assumed to be constant along the beam section, i.e. the axis of the bent beam forms an arc having radius $1/\kappa$. Moreover, in what follows it will be assumed that the curvature κ is small (in comparison with the inverse of a typical length scale of the cross section). Due to the bending the upper part of the beam is stretched, whereas the bottom part is compressed. The surface separating the stretched part from the compressed part is called the neutral surface. In Fig. 8.1 the neutral surface is assumed to coincides with the $y = 0$ plane in the limit of $\kappa \to 0$. Thus, in the same limit of $\kappa \to 0$ the strain ϵ (which is positive in the stretch part and negative in the compressed part) at any point in the bent beam is proportional to y, and it is given by

$$\epsilon = \kappa y \; . \tag{8.1}$$

Let M be the bending moment calculated with respect to the left end of the beam's axis (see Fig. 8.1). The bending moment is found by integrating over the beam's cross section

$$M = \int \mathrm{d}A \; y\sigma \; , \tag{8.2}$$

where σ is the normal stress. By using the relation $\sigma = E\epsilon$ [see Eq. (7.1)], where E is the Young's modulus, one finds that

$$M = EI\kappa \; , \tag{8.3}$$

where I, which is given by

$$I = \int \mathrm{d}A \; y^2 \; , \tag{8.4}$$

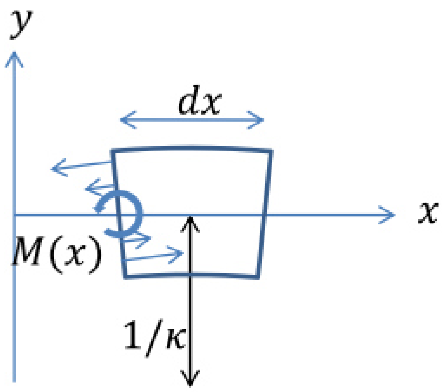


Fig. 8.1. The bending moment.

is called the moment of inertia. Note that I depends on the chosen plane of bending (which is xy in the example above). The term EI represents the flexural rigidity of the beam in the bending plane (the xy plane in the present case).

Exercise 8.1.1. Calculate the moment of inertia I for a circular cross section of radius a.

Solution 8.1.1. According to Eq. (8.4) I is given by

$$I = \int_{-a}^{a} \mathrm{d}z \int_{-\sqrt{a^2-z^2}}^{\sqrt{a^2-z^2}} \mathrm{d}y\, y^2 = \frac{2}{3} \int_{-a}^{a} \mathrm{d}z\, \left(a^2 - z^2\right)^{3/2} = \frac{\pi a^4}{4} . \tag{8.5}$$

Exercise 8.1.2. Calculate I for a rectangular cross section having width w in the y direction and thickness t in the z direction. Consider both cases of bending in the xy plane and in xz plane.

Solution 8.1.2. For the case of bending in the xy plane I is given by

$$I = \int_{-\frac{t}{2}}^{\frac{t}{2}} \mathrm{d}z \int_{-\frac{w}{2}}^{\frac{w}{2}} \mathrm{d}y\, y^2 = \frac{w^3 t}{12} . \tag{8.6}$$

Similarly, for the case of bending in the xz plane I is given by $wt^3/12$.

What is the energy u_{b} needed to bend a short beam of length $\mathrm{d}x$ into an arc of curvature κ? The twisting angle of the beam θ, i.e. the angle between the axis of the beam at point x and the axis at point $x + \mathrm{d}x$, is given by

$$\theta = \kappa \mathrm{d}x . \tag{8.7}$$

The energy u_{b} is found by integrating the work elements $M\mathrm{d}\theta$ [see Eq. (8.3)]

$$u_{\mathrm{b}} = \int_0^{\kappa \mathrm{d}x} M \,\mathrm{d}\theta = \frac{EI}{\mathrm{d}x}\int_0^{\kappa \mathrm{d}x} \theta \,\mathrm{d}\theta = \frac{EI\kappa^2}{2}\mathrm{d}x \;, \tag{8.8}$$

thus the energy per unit length is given by

$$\frac{u_{\mathrm{b}}}{\mathrm{d}x} = \frac{EI\kappa^2}{2} \;. \tag{8.9}$$

As was noted above, for a curve having the shape of an arc of radius R the curvature κ is given by $\kappa = 1/R$. What is the curvature of a general planar curve $\mathbf{r}(s) = (x(s), y(s))$? The definition of the curvature is relatively simple provided that the curve $\mathbf{r}(s)$ is given in what is called arc-length parametrization. For this case the parameter s measures the length along the curve. In other words, for a curve given in arc-length parametrization the following holds $|\mathrm{d}\mathbf{r}/\mathrm{d}s| = 1$. The curvature κ of a curve that is given in arc-length parametrization is defined as $\kappa = \left|\mathrm{d}^2\mathbf{r}/\mathrm{d}s^2\right|$.

Exercise 8.1.3. Find an expression for the curvature κ of a general planar curve $\mathbf{r}(\xi) = (x(\xi), y(\xi))$, for which the parametrization is not necessarily arc-length parametrization.

Solution 8.1.3. The parameter s, which is defined by

$$s(\xi) = \int_{\xi_0}^{\xi} \mathrm{d}\xi' \left|\frac{\mathrm{d}\mathbf{r}}{\mathrm{d}\xi}\right| \;, \tag{8.10}$$

represents the length of the curve $\mathbf{r}(\xi)$ from ξ_0 to ξ, and thus can be used for arc-length parametrization. The normalized vector $\mathrm{d}\mathbf{r}/\mathrm{d}s$ can be expressed in terms of the angle θ

$$\frac{\mathrm{d}\mathbf{r}}{\mathrm{d}s} = \left(\frac{\mathrm{d}x}{\mathrm{d}s}, \frac{\mathrm{d}y}{\mathrm{d}s}\right) = (\cos\theta, \sin\theta) \;, \tag{8.11}$$

where

$$\theta = \tan^{-1}\frac{\mathrm{d}y/\mathrm{d}s}{\mathrm{d}x/\mathrm{d}s} = \tan^{-1}\frac{\mathrm{d}y/\mathrm{d}\xi}{\mathrm{d}x/\mathrm{d}\xi} \;. \tag{8.12}$$

The curvature κ is defined by

$$\kappa = \left|\frac{\mathrm{d}^2\mathbf{r}}{\mathrm{d}s^2}\right| \;, \tag{8.13}$$

thus

$$\kappa = |(-\sin\theta, \cos\theta)| \left|\frac{\mathrm{d}\theta}{\mathrm{d}s}\right| = \left|\frac{\mathrm{d}\theta}{\mathrm{d}s}\right| \;. \tag{8.14}$$

or

$$\kappa = \left|\frac{\mathrm{d}\theta}{\mathrm{d}s}\right| = \left|\frac{\mathrm{d}\xi}{\mathrm{d}s}\frac{\mathrm{d}\theta}{\mathrm{d}\xi}\right| = \left|\frac{\mathrm{d}\xi}{\mathrm{d}s}\frac{\mathrm{d}\tan^{-1}\frac{\mathrm{d}y/\mathrm{d}\xi}{\mathrm{d}x/\mathrm{d}\xi}}{\mathrm{d}\xi}\right| , \tag{8.15}$$

and therefore with the help of the general identity

$$\frac{\mathrm{d}\tan^{-1} a}{\mathrm{d}a} = \frac{1}{1+a^2} , \tag{8.16}$$

and the relation

$$\left|\frac{\mathrm{d}s}{\mathrm{d}\xi}\right| = \sqrt{\left(\frac{\mathrm{d}x}{\mathrm{d}\xi}\right)^2 + \left(\frac{\mathrm{d}y}{\mathrm{d}\xi}\right)^2} , \tag{8.17}$$

one finds that

$$\kappa = \frac{\left|\frac{\mathrm{d}x}{\mathrm{d}\xi}\frac{\mathrm{d}^2y}{\mathrm{d}\xi^2} - \frac{\mathrm{d}^2x}{\mathrm{d}\xi^2}\frac{\mathrm{d}y}{\mathrm{d}\xi}\right|}{\left[\left(\frac{\mathrm{d}x}{\mathrm{d}\xi}\right)^2 + \left(\frac{\mathrm{d}y}{\mathrm{d}\xi}\right)^2\right]^{3/2}} . \tag{8.18}$$

The last result (8.18) can be used to find the curvature of a planar curve given in the form of the function $y(x)$. For this case the coordinate x plays the role of the parameter ξ and the curvature is given by

$$\kappa = \frac{\left|\frac{\mathrm{d}^2y}{\mathrm{d}x^2}\right|}{\left[1 + \left(\frac{\mathrm{d}y}{\mathrm{d}x}\right)^2\right]^{3/2}} . \tag{8.19}$$

According to the geometrical definition of the curvature, i.e. $\kappa = \left|\mathrm{d}^2\mathbf{r}/\mathrm{d}s^2\right|$, κ is nonnegative for both concave and convex curves. To be able to distinguish between these two cases a revised definition, according to which the curvature is given by $\kappa = -\mathrm{d}^2\mathbf{r}/\mathrm{d}s^2$, will be employed below. According to the revised definition Eq. (8.19) becomes

$$\kappa = -\frac{\frac{\mathrm{d}^2y}{\mathrm{d}x^2}}{\left[1 + \left(\frac{\mathrm{d}y}{\mathrm{d}x}\right)^2\right]^{3/2}} . \tag{8.20}$$

As can be seen by examining Fig. 8.1 and Eq. (8.3), the revised definition is consistent with the way the circular arrow in Fig. 8.1 is drawn, which points in the direction corresponding to the case where M is positive.

8.2 Lagrangian

Consider a beam made of a material having mass density ρ and Young's modulus E. In the absent of tension the length of the beam is l_0. The beam is doubly clamped to a substrate at the points $(x, y) = (0, \pm l/2)$ and the motion of the beam's axis, which is described by the height function $y(x, t)$, is assumed to be exclusively in the xy plane. The total length of the beam is $l + \Delta l$, where Δl is the change in the length of the beam due to the deflection, which is found by integrating the length of infinitesimal sections $\sqrt{(\mathrm{d}x)^2 + (\mathrm{d}y)^2} = \mathrm{d}x\left(\sqrt{1 + (\partial y/\partial x)^2}\right)$, i.e.

$$\Delta l = \int_{-l/2}^{l/2} \mathrm{d}x \left(\sqrt{1 + \left(\frac{\partial y}{\partial x}\right)^2} - 1\right) . \tag{8.21}$$

When $\partial y/\partial x \ll 1$ one has

$$\Delta l \simeq \frac{1}{2} \int_{-l/2}^{l/2} \mathrm{d}x \left(\frac{\partial y}{\partial x}\right)^2 . \tag{8.22}$$

The change in the elastic energy U_{e} due to beam's elongation is found by integrating over the force $EA_{\mathrm{cs}} \times \Delta l/l_0$ [see Eq. (7.1)], where A_{cs} is the cross section area of the beam, i.e.

$$\begin{aligned} U_{\mathrm{e}} &= EA_{\mathrm{cs}} \int_l^{l+\Delta l} \mathrm{d}l' \, \frac{l' - l_0}{l_0} \\ &= EA_{\mathrm{cs}} \frac{l - l_0}{l_0} \Delta l + \frac{EA_{\mathrm{cs}}}{2l_0} (\Delta l)^2 \\ &\simeq EA_{\mathrm{cs}} \frac{l - l_0}{l_0} \Delta l + \frac{EA_{\mathrm{cs}}}{2l} (\Delta l)^2 \\ &= N\Delta l + \frac{EA_{\mathrm{cs}}}{2l} (\Delta l)^2 , \end{aligned} \tag{8.23}$$

where

$$N = EA_{\mathrm{cs}} \frac{l - l_0}{l_0} \tag{8.24}$$

is the tension in the beam for the straight case where $y = 0$.

The bending energy U_{b} is found with the help of Eqs. (8.9) and (8.20)

$$U_{\mathrm{b}} = \frac{EI}{2} \int_{-l/2}^{l/2} \mathrm{d}x \, \kappa^2 = \frac{EI}{2} \int_{-l/2}^{l/2} \mathrm{d}x \, \frac{\left(\frac{\partial^2 y}{\partial x^2}\right)^2}{\left[1 + \left(\frac{\partial y}{\partial x}\right)^2\right]^3} . \tag{8.25}$$

For the case where $\partial y/\partial x \ll 1$ one has

$$U_{\mathrm{b}} = \frac{EI}{2} \int_{-l/2}^{l/2} \mathrm{d}x \, \left(\frac{\partial^2 y}{\partial x^2}\right)^2 . \tag{8.26}$$

The kinetic energy T_{k} is given by

$$T_{\mathrm{k}} = \frac{\rho A_{\mathrm{cs}}}{2} \int_{-l/2}^{l/2} \mathrm{d}x \, \left(\frac{\partial y}{\partial t}\right)^2 . \tag{8.27}$$

In addition, the potential energy U_{f} due to an externally applied force f per unit length acting in the y direction is given by

$$U_{\mathrm{f}} = -\int_{-l/2}^{l/2} \mathrm{d}x \, fy . \tag{8.28}$$

The Lagrangian functional $\mathcal{L}$, which is given by

$$\mathcal{L} = T_{\mathrm{k}} - U_{\mathrm{e}} - U_{\mathrm{b}} - U_{\mathrm{f}} , \tag{8.29}$$

is expressed in terms of the height function $y(x,t)$ as

$$\mathcal{L} = \int_{-l/2}^{l/2} \mathrm{d}x \, L - \frac{A_{\mathrm{cs}} E}{8l} \left(\int_{-l/2}^{l/2} \mathrm{d}x \left(\frac{\partial y}{\partial x}\right)^2 \right)^2 , \tag{8.30}$$

where

$$L = \frac{A_{\mathrm{cs}}\rho}{2} \left(\frac{\partial y}{\partial t}\right)^2 - \frac{N}{2} \left(\frac{\partial y}{\partial x}\right)^2 - \frac{EI}{2} \left(\frac{\partial^2 y}{\partial x^2}\right)^2 + fy \tag{8.31}$$

is the Lagrangian density.

8.3 Boundary Conditions

Some common boundary conditions are depicted in Fig. 8.2.

8.4 Equation of Motion

The beam's equation of motion is found by evaluating the variation in the action δS due to infinitesimal change δy is the height function. First, consider the contribution to δS due to the linear part of $\mathcal{L}$ [first term in Eq. (8.30)].

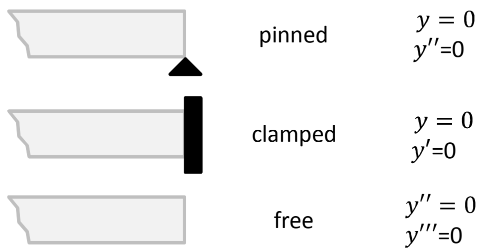


Fig. 8.2. Typical types of boundary conditions.

Exercise 8.4.1. Consider the general case, where the Lagrangian $\mathcal{L}$ can be expressed as

$$\mathcal{L} = \int_{-l/2}^{l/2} \mathrm{d}x \; L \; , \tag{8.32}$$

and where L, which is generally called the *Lagrangian density*, is allowed to depend on y, $\partial y/\partial t$, $\partial y/\partial x$ and on $\partial^2 y/\partial x^2$. The boundary conditions at the end points $x = -l/2$ and $x = l/2$ are taken to be given by $y = 0$ and $\partial y/\partial x = 0$, i.e. the beam is assumed to be doubly clamped. Find the corresponding equation of motion.

Solution 8.4.1. To lowest nonvanishing order in δy the variation in the action δS can be expressed as

$$\delta S = \int_{t_1}^{t_2} \mathrm{d}t \int_{-l/2}^{l/2} \mathrm{d}x \, \left(\delta y \frac{\partial L}{\partial y} + \frac{\partial\left(\delta y\right)}{\partial t} \frac{\partial L}{\partial \frac{\partial y}{\partial t}} + \frac{\partial\left(\delta y\right)}{\partial x} \frac{\partial L}{\partial \frac{\partial y}{\partial x}} + \frac{\partial^2\left(\delta y\right)}{\partial x^2} \frac{\partial L}{\partial \frac{\partial^2 y}{\partial x^2}} \right) \, . \tag{8.33}$$

The second term in Eq. (8.33) can be evaluated by performing the integration over t by parts [recall that $\delta y\left(x, t_1\right) = \delta y\left(x, t_2\right) = 0$]

$$\int_{t_1}^{t_2} \mathrm{d}t \int_{-l/2}^{l/2} \mathrm{d}x \, \frac{\partial\left(\delta y\right)}{\partial t} \frac{\partial L}{\partial \frac{\partial y}{\partial t}} = -\int_{t_1}^{t_2} \mathrm{d}t \int_{-l/2}^{l/2} \mathrm{d}x \; \delta y \frac{\partial}{\partial t} \frac{\partial L}{\partial \frac{\partial y}{\partial t}} \; . \tag{8.34}$$

The third term in Eq. (8.33) can be evaluated by performing the integration over x by parts (recall the boundary condition $y = 0$ at he end points $x = -l/2$ and $x = l/2$)

$$\int_{t_1}^{t_2} \mathrm{d}t \int_{-l/2}^{l/2} \mathrm{d}x \, \frac{\partial\left(\delta y\right)}{\partial x} \frac{\partial L}{\partial \frac{\partial y}{\partial x}} = -\int_{t_1}^{t_2} \mathrm{d}t \int_{-l/2}^{l/2} \mathrm{d}x \; \delta y \frac{\partial}{\partial x} \frac{\partial L}{\partial \frac{\partial y}{\partial x}} \; . \tag{8.35}$$

The forth term in Eq. (8.33) can be evaluated by performing the integration over x by parts twice (recall the boundary conditions $y = 0$ and $\partial y/\partial x = 0$ at the end points $x = -l/2$ and $x = l/2$)

$$\begin{aligned}\int_{t_1}^{t_2} \mathrm{d}t \int_{-l/2}^{l/2} \mathrm{d}x\, \frac{\partial^2 (\delta y)}{\partial x^2} \frac{\partial L}{\partial \frac{\partial^2 y}{\partial x^2}} &= -\int_{t_1}^{t_2} \mathrm{d}t \int_{-l/2}^{l/2} \mathrm{d}x\, \frac{\partial (\delta y)}{\partial x} \frac{\partial}{\partial x} \frac{\partial L}{\partial \frac{\partial^2 y}{\partial x^2}} \\ &= \int_{t_1}^{t_2} \mathrm{d}t \int_{-l/2}^{l/2} \mathrm{d}x\, \delta y \frac{\partial^2}{\partial x^2} \frac{\partial L}{\partial \frac{\partial^2 y}{\partial x^2}} \ .\end{aligned} \tag{8.36}$$

By combining these results one finds that

$$\delta S = \int_{t_1}^{t_2} \mathrm{d}t \int_{-l/2}^{l/2} \mathrm{d}x\, \left(\frac{\partial L}{\partial y} - \frac{\partial}{\partial t} \frac{\partial L}{\partial \frac{\partial y}{\partial t}} - \frac{\partial}{\partial x} \frac{\partial L}{\partial \frac{\partial y}{\partial x}} + \frac{\partial^2}{\partial x^2} \frac{\partial L}{\partial \frac{\partial^2 y}{\partial x^2}} \right) \delta y \ , \tag{8.37}$$

thus, according to the least action principle the following must holds

$$0 = \frac{\partial L}{\partial y} - \frac{\partial}{\partial t} \frac{\partial L}{\partial \frac{\partial y}{\partial t}} - \frac{\partial}{\partial x} \frac{\partial L}{\partial \frac{\partial y}{\partial x}} + \frac{\partial^2}{\partial x^2} \frac{\partial L}{\partial \frac{\partial^2 y}{\partial x^2}} \ . \tag{8.38}$$

The contribution due to the nonlinear part of $\mathcal{L}$ [second term in Eq. (8.30)] is found using the following relation

$$
\begin{aligned}
&\delta \left(\int_{-l/2}^{l/2} \mathrm{d}x \left(\frac{\partial y}{\partial x} \right)^2 \right)^2 \\
&= \left(\int_{-l/2}^{l/2} \mathrm{d}x \left(\frac{\partial y}{\partial x} + \frac{\partial (\delta y)}{\partial x} \right)^2 \right)^2 - \left(\int_{-l/2}^{l/2} \mathrm{d}x \left(\frac{\partial y}{\partial x} \right)^2 \right)^2 \\
&= \int_{-l/2}^{l/2} \mathrm{d}x \left[\left(\frac{\partial y}{\partial x} + \frac{\partial (\delta y)}{\partial x} \right)^2 - \left(\frac{\partial y}{\partial x} \right)^2 \right] \\
&\times \int_{-l/2}^{l/2} \mathrm{d}x \left[\left(\frac{\partial y}{\partial x} + \frac{\partial (\delta y)}{\partial x} \right)^2 + \left(\frac{\partial y}{\partial x} \right)^2 \right] \\
&\simeq 4 \left(\int_{-l/2}^{l/2} \mathrm{d}x \frac{\partial y}{\partial x} \frac{\partial (\delta y)}{\partial x} \right) \left(\int_{-l/2}^{l/2} \mathrm{d}x \left(\frac{\partial y}{\partial x} \right)^2 \right) \\
&= -4 \int_{-l/2}^{l/2} \mathrm{d}x \frac{\partial^2 y}{\partial x^2} \delta y \int_{-l/2}^{l/2} \mathrm{d}x \left(\frac{\partial y}{\partial x} \right)^2 .
\end{aligned}
\tag{8.39}
$$

where integration by parts has been performed in the last step. By combining the linear [see Eq. (8.37)] and nonlinear contributions one finds that

$$
\begin{aligned}
\delta S = &\int_{t_1}^{t_2} \mathrm{d}t \int_{-l/2}^{l/2} \mathrm{d}x \; \delta y \left[f - A_{\mathrm{cs}} \rho \frac{\partial^2 y}{\partial t^2} - EI \frac{\partial^4 y}{\partial x^4} \right. \\
&\left. + \left(N + \frac{A_{\mathrm{cs}} E}{2l} \int_{-l/2}^{l/2} \mathrm{d}x \left(\frac{\partial y}{\partial x} \right)^2 \right) \frac{\partial^2 y}{\partial x^2} \right] ,
\end{aligned}
\tag{8.40}
$$

thus, according to the least action principle the following must holds

$$
\begin{aligned}
\Upsilon \frac{\partial^2 y}{\partial t^2} = &\left(N + \frac{A_{\mathrm{cs}} E}{2l} \int_{-l/2}^{l/2} \mathrm{d}x \left(\frac{\partial y}{\partial x} \right)^2 \right) \frac{\partial^2 y}{\partial x^2} \\
&- EI \frac{\partial^4 y}{\partial x^4} + f ,
\end{aligned}
\tag{8.41}
$$

where $\Upsilon = \rho A_{\mathrm{cs}}$ is the mass density per unit length.

8.5 Consistency with Newton's Second Law

The equation of motion (8.41) can be rewritten as

$$\Upsilon \frac{\partial^2 y}{\partial t^2} = f_{\mathrm{T}} + f_{\mathrm{B}} + f \; , \tag{8.42}$$

where

$$f_{\mathrm{T}} = N_{\mathrm{t}} \frac{\partial^2 y}{\partial x^2} \; , \tag{8.43}$$

$$f_{\mathrm{B}} = -EI \frac{\partial^4 y}{\partial x^4} \; , \tag{8.44}$$

and where

$$N_{\mathrm{t}} = N + \frac{A_{\mathrm{cs}} E}{l} \frac{1}{2} \int_{-l/2}^{l/2} \mathrm{d}x \left(\frac{\partial y}{\partial x} \right)^2 \; . \tag{8.45}$$

Note that [see Eq. (8.22)]

$$N_{\mathrm{t}} = N + E A_{\mathrm{cs}} \frac{\Delta l}{l} \; , \tag{8.46}$$

thus N_{t} is the total tension in the beam.

The equation of motion (8.42) is consistent with Newton's second law. To see this consider the infinitesimal section of the beam from x to $x + \mathrm{d}x$.

Claim. The force due to tension acting on the infinitesimal section is $f_{\mathrm{T}} \mathrm{d}x$.

Proof. As can be seen from Fig. 8.3, the force due to tension acting on this section in the y direction is approximately given for the case where $y' = \partial y / \partial x \ll 1$ by

$$N_{\mathrm{t}} y' (x + \mathrm{d}x) - N_{\mathrm{t}} y' (x) \simeq N_{\mathrm{t}} y'' \mathrm{d}x = f_{\mathrm{T}} \mathrm{d}x \; . \tag{8.47}$$

where $y'' = \partial^2 y / \partial x^2$.

Claim. The bending force acting on the infinitesimal section is $f_{\mathrm{B}} \mathrm{d}x$.

Proof. Consider the infinitesimal section, which is shown in Fig. 8.4. The figure shows the bending moments $M(x)$ and $M(x + \mathrm{d}x)$ and the shearing forces $F_{\mathrm{t}}(x)$ and $F_{\mathrm{t}}(x + \mathrm{d}x)$ acting on both ends of the short section at x

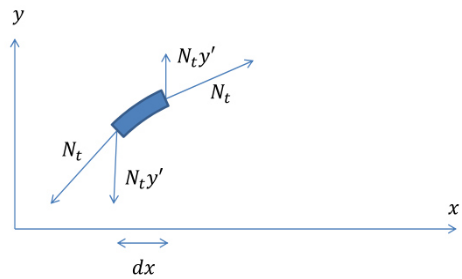


Fig. 8.3. The force acting on a small section of the beam due to tension.

and at $x + \mathrm{d}x$. The total bending force F_y acting on the short beam in the y direction is given by

$$F_y = F_\mathrm{t}(x + \mathrm{d}x) - F_\mathrm{t}(x) = F_\mathrm{t}' \mathrm{d}x \; . \tag{8.48}$$

Let M_0 be the total moment acting on the short section with respect to the left end of the beam's axis (see Fig. 8.4). The requirement that M_0 vanishes leads to

$$0 = M(x) - M(x + \mathrm{d}x) + F_\mathrm{t}(x + \mathrm{d}x)\,\mathrm{d}x \; , \tag{8.49}$$

and therefore

$$F_\mathrm{t} = M' \; . \tag{8.50}$$

Combining the last result with Eq. (8.48) yields

$$F_y = M'' \mathrm{d}x \; . \tag{8.51}$$

For the case where $y' \ll 1$ the curvature κ is approximately given by $\kappa = -y''$ [see Eq. (8.20)]. Thus the force F_y can be expressed as [see Eq. (8.3)]

$$F_y = -EIy'''' \mathrm{d}x = f_\mathrm{B} \mathrm{d}x \; . \tag{8.52}$$

The above results (8.47) and (8.52) show that indeed Eq. (8.42) is consistent with Newton's second law.

8.6 String

In the case of a string, the stiffness term proportional to EI in Eq. (8.41) is considered as negligibly small. In this limit Eq. (8.41) becomes

$$\Upsilon \frac{\partial^2 y}{\partial t^2} = N_\mathrm{t} \frac{\partial^2 y}{\partial x^2} + f \; . \tag{8.53}$$

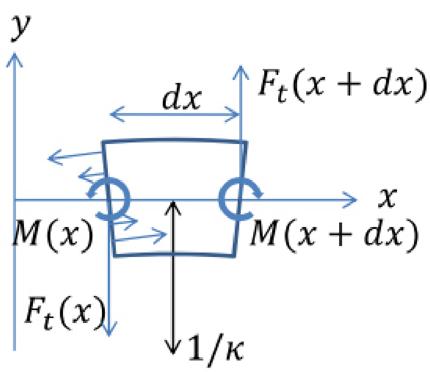


Fig. 8.4. A bent beam having curvature κ.

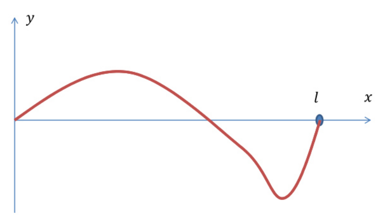


Fig. 8.5. Vibrating string.

In this section the case where the deflection is small will be considered. As can be seen form Eq. (8.45), for that case one can employ the approximation $N_t \simeq N$. In this approximation the wave equation for the string becomes linear. The boundary conditions at the end points $x = 0$ and $x = l$ are taken to be given by

$$y(0,t) = y(l,t) = 0 \; . \tag{8.54}$$

8.6.1 Normal Modes

Consider a solution to the wave equation (8.53) having the form

$$y(x,t) = c(t)\,\mathcal{Y}(x) \; , \tag{8.55}$$

where $c(t)$ depends on time only and where $\mathcal{Y}(x)$ depends on x only. Substituting into Eq. (8.53) yields

$$\frac{\Upsilon}{N}\frac{\ddot{c}}{c} = \frac{\mathcal{Y}''}{\mathcal{Y}} \; . \tag{8.56}$$

where overdot denotes derivative with respect to time and where prime denotes derivative with respect to x. While the left hand side is a function of t only, the right hand side is a function of x only, and therefore both must equal a constant, which is denoted by $-k^2$, i.e. the following holds

$$\mathcal{Y}'' = -k^2 \mathcal{Y} , \tag{8.57}$$

and

$$\ddot{c} = -\frac{N k^2}{\Upsilon} c . \tag{8.58}$$

A general solution of Eq. (8.57) has the form $A_1 \sin(kx) + A_2 \cos(kx)$, where both A_1 and A_2 are constants. The boundary conditions $\mathcal{Y}(0) = \mathcal{Y}(l) = 0$ lead to the requirements that $A_2 = 0$ and $kl = n\pi$, where n is integer. To ensures normalization (as will be seen below) the constant A_1 is taken to be given by $A_1 = \sqrt{2/l}$, and thus the nth solution is given by

$$\mathcal{Y}_n(x) = \sqrt{\frac{2}{l}} \sin(k_n x) , \tag{8.59}$$

where

$$k_n = \frac{\pi n}{l} . \tag{8.60}$$

Exercise 8.6.1. Calculate the inner product $\langle \mathcal{Y}_n | \mathcal{Y}_m \rangle$, which is is defined by

$$\langle \mathcal{Y}_n | \mathcal{Y}_m \rangle = \int_0^l \mathrm{d}x\, \mathcal{Y}_n(x) \mathcal{Y}_m(x) . \tag{8.61}$$

Solution 8.6.1. The following holds

$$\begin{aligned} \int_0^l \mathrm{d}x\, \mathcal{Y}_n(x) \mathcal{Y}_m(x) &= \frac{2}{l} \int_0^l \mathrm{d}x\, \sin\frac{\pi n x}{l} \sin\frac{\pi m x}{l} \\ &= \frac{1}{l} \int_0^l \mathrm{d}x \left(\cos\frac{\pi(n-m)x}{l} - \cos\frac{\pi(n+m)x}{l} \right) \\ &= \frac{\sin\frac{\pi(n-m)x}{l}}{\pi(n-m)} - \frac{\sin\frac{\pi(n+m)x}{l}}{\pi(n+m)} , \end{aligned} \tag{8.62}$$

thus

$$\int_0^l \mathrm{d}x\, \mathcal{Y}_n(x) \mathcal{Y}_m(x) = \delta_{n,m} . \tag{8.63}$$

Exercise 8.6.2. Calculate the inner product $\langle \mathcal{Y}_n' | \mathcal{Y}_m' \rangle$, where $\mathcal{Y}_n' = \mathrm{d}\mathcal{Y}_n/\mathrm{d}x$.

Solution 8.6.2. Using integration by parts one finds that [recall that $\mathcal{Y}_n(0) = \mathcal{Y}_n(l) = 0$]

$$\begin{aligned}
\langle \mathcal{Y}_n' | \mathcal{Y}_m' \rangle &= \int_0^l \mathrm{d}x\, \mathcal{Y}_n' \mathcal{Y}_m' \\
&= -\int_0^l \mathrm{d}x\, \mathcal{Y}_n \mathcal{Y}_m'' \\
&= k_m^2 \int_0^l \mathrm{d}x\, \mathcal{Y}_n \mathcal{Y}_m \\
&= k_m^2 \delta_{n,m} \; .
\end{aligned} \tag{8.64}$$

The set of orthonormal functions $\{\mathcal{Y}_n(x)\}_n$ are the solutions of a Sturm–Liouville problem (i.e. linear boundary conditions problem), and thus it can serve as a complete basis, with which the general solution $y(x,t)$ can be expanded as

$$y(x,t) = \sum_{n=1}^{\infty} c_n(t)\, \mathcal{Y}_n(x) \; . \tag{8.65}$$

Substituting this expansion into the Lagrangian (8.30), which for the limit of a string becomes (in the absence of externally applied force)

$$\mathcal{L} = \int_{-l/2}^{l/2} \mathrm{d}x \left[\frac{\Upsilon}{2} \left(\frac{\partial y}{\partial t} \right)^2 - \frac{N}{2} \left(\frac{\partial y}{\partial x} \right)^2 \right] , \tag{8.66}$$

yields

$$\mathcal{L} = \frac{\Upsilon}{2} \sum_{n,m=1}^{\infty} \dot{c}_n \dot{c}_m \int_0^l \mathrm{d}x\, \mathcal{Y}_n \mathcal{Y}_m - \frac{N}{2} \sum_{n,m=1}^{\infty} c_n c_m \int_0^l \mathrm{d}x\, \mathcal{Y}_n' \mathcal{Y}_m' \; , \tag{8.67}$$

thus with the help of Eqs. (8.63) and (8.64) one finds that

$$\mathcal{L} = \sum_{n=1}^{\infty} \left(\frac{\Upsilon \dot{c}_n^2}{2} - \frac{N k_n^2 c_n^2}{2} \right) . \tag{8.68}$$

The Euler-Lagrange set of equations (1.8) for the coordinates c_n is given by

$$\ddot{c}_n = -\omega_n^2 c_n \ , \tag{8.69}$$

where

$$\omega_n = \sqrt{\frac{N}{\Upsilon}} k_n = \sqrt{\frac{N}{\Upsilon}} \frac{\pi n}{l} \ . \tag{8.70}$$

Note that these equations have the same form as Eq. (8.58), which is the form of an equation of motion of an undamped harmonic oscillator [see Eq. (3.2)]. Thus the nth normal mode has the same dynamics as an harmonic oscillator having angular resonance frequency ω_n. The corresponding resonance frequency $f_n = \omega_n / 2\pi$ is given by

$$f_n = \sqrt{\frac{N}{\Upsilon}} \frac{n}{2l} \ . \tag{8.71}$$

8.7 The Tension Free Case

In the tension-free case, i.e. the case for which $N = 0$, Eq. (8.41) becomes

$$\Upsilon \frac{\partial^2 y}{\partial t^2} = -EI \frac{\partial^4 y}{\partial x^4} + f \ . \tag{8.72}$$

Substituting a solution having the form

$$y(x,t) = \exp(kx - i\omega t) \tag{8.73}$$

yields

$$\omega^2 \Upsilon = EIk^4 \ , \tag{8.74}$$

thus the four possible solutions are $k = \mathcal{K}$, $-\mathcal{K}$, $i\mathcal{K}$ and $-i\mathcal{K}$, where

$$\mathcal{K} = \left(\frac{\omega^2 \Upsilon}{EI} \right)^{1/4} \ , \tag{8.75}$$

and thus the general solution can be expressed as

$$y(x,t) = e^{-i\omega t} \left[A \cosh(\mathcal{K}x) + B \sinh(\mathcal{K}x) + C \cos(\mathcal{K}x) + D \sin(\mathcal{K}x) \right] \ . \tag{8.76}$$

As an example consider the case of a *cantilever*. For this case one end of the beam is assumed to be clamped and the other free. The corresponding boundary conditions are taken to be $y = 0$ and $y' = 0$ at $x = 0$ and $y'' = 0$ and $y''' = 0$ at $x = l$. Note that the boundary conditions at the free end at $x = l$ express the requirements that both bending moment $M = -EIy''$ and shearing force $F_{\mathrm{t}} = -EIy'''$ vanish.

The boundary conditions at the clamped end at $x = 0$ are satisfied provided that $C = -A$ and $D = -B$. For this case $y(x,t)$ is given by

$$y(x,t) = e^{-i\omega t}\left[A\left(\cosh(\mathcal{K}x) - \cos(\mathcal{K}x)\right) + B\left(\sinh(\mathcal{K}x) - \sin(\mathcal{K}x)\right)\right] . \tag{8.77}$$

The boundary conditions at $x = l$ yield

$$\begin{pmatrix} \cosh(\mathcal{K}l) + \cos(\mathcal{K}l) & \sinh(\mathcal{K}l) + \sin(\mathcal{K}l) \\ \sinh(\mathcal{K}l) - \sin(\mathcal{K}l) & \cosh(\mathcal{K}l) + \cos(\mathcal{K}l) \end{pmatrix} \begin{pmatrix} A \\ B \end{pmatrix} = 0 . \tag{8.78}$$

Nontrivial solution exists provided that the determinant of the matrix above vanishes, i.e. provided that

$$\cosh(\mathcal{K}l)\cos(\mathcal{K}l) = -1 . \tag{8.79}$$

The solutions can be labeled in increasing order as $\mathcal{K}l = a_n$, where $a_1 = 1.8751$, $a_2 = 4.6941$, $a_3 = 7.8548$, etc. The corresponding resonance frequencies $f_n = \omega_n/2\pi$ are given by [see Eq. (8.75)]

$$f_n = \frac{a_n^2}{2\pi}\frac{1}{l^2}\sqrt{\frac{EI}{\Upsilon}} . \tag{8.80}$$

The frequency of the fundamental mode is given by

$$f_n = \frac{0.55959}{l^2}\sqrt{\frac{EI}{\Upsilon}} , \tag{8.81}$$

and the following holds $f_2/f_1 = 6.266\,9$ and $f_3/f_1 = 17.548$.

8.8 Buckling

While for taut beam the tension $N > 0$, here we consider the case where the tension N is negative, i.e. the case where compressive stress is applied to the beam. For that case we will use the notation $P = -N$, where P is assumed to be nonnegative. We first consider the case where the deflection is small, for which the approximation $N_{\mathrm{t}} \simeq N$ can be employed [see Eq. (8.45)]. The equation of motion (8.41) for the static case, i.e. the case for which $\partial y/\partial t = 0$, and in the absent of externally applied force, is given by

$$0 = \frac{P}{EI}\frac{\partial^2 y}{\partial x^2} + \frac{\partial^4 y}{\partial x^4} . \tag{8.82}$$

The general homogeneous solution is given by

$$y(x) = A + Bx + C\sin\left(\sqrt{\frac{P}{EI}}x\right) + D\cos\left(\sqrt{\frac{P}{EI}}x\right) , \tag{8.83}$$

where A, B, C and D are constants. Consider first the case of *pinned* boundary conditions at both ends of the beam, which are taken to be $x = 0$ and $x = l$, i.e.

$$y(0) = y(l) = 0 , \tag{8.84}$$

$$y''(0) = y''(l) = 0 . \tag{8.85}$$

These boundary conditions yield

$$A + D = A + Bl + C \sin\left(\sqrt{\frac{P}{EI}} l\right) + D \cos\left(\sqrt{\frac{P}{EI}} l\right) = 0 , \tag{8.86}$$

$$D = C \sin\left(\sqrt{\frac{P}{EI}} l\right) + D \cos\left(\sqrt{\frac{P}{EI}} l\right) = 0 , \tag{8.87}$$

and therefore $A = B = D = 0$. Nontrivial solution is possible (i.e. solution for which $C \neq 0$) provided that

$$\sqrt{\frac{P}{EI}} l = n\pi . \tag{8.88}$$

where n is integer. Critical load of the nth nontrivial solution is given by

$$P_n = \frac{n^2 \pi^2 EI}{l^2} , \tag{8.89}$$

and the corresponding mode shape is given by

$$y_n(x) = C \sin \frac{n\pi x}{l} . \tag{8.90}$$

Above the lowest buckling load, which for the present case of doubly pinned beam is given by $P_1 = \pi^2 EI/l^2$, the beam is expected to buckle. As will be demonstrated in the next exercise, the buckling load depends on the boundary conditions.

Exercise 8.8.1. Find the first critical load and the corresponding mode shape of a beam for the case of doubly clamped boundary conditions.

Solution 8.8.1. For this case the boundary conditions are

$$y(0) = y(l) = 0 , \tag{8.91}$$

$$y'(0) = y'(l) = 0 . \tag{8.92}$$

In matrix form these conditions can be expressed as

$$M \begin{pmatrix} A \\ B \\ C \\ D \end{pmatrix} = 0 . \tag{8.93}$$

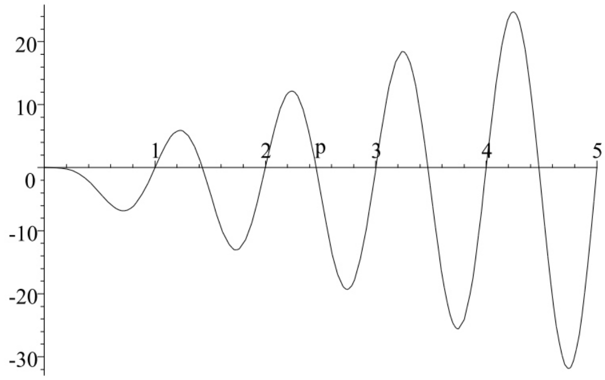


Fig. 8.6. The function $\mathcal{D}(p) = 2\pi p \sin(2\pi p) - 2 + 2\cos(2\pi p)$.

where the matrix M is given by

$$M = \begin{pmatrix} 1 & 0 & 0 & 1 \\ 1 & l & \sin\left(\sqrt{\frac{P}{EI}}l\right) & \cos\left(\sqrt{\frac{P}{EI}}l\right) \\ 0 & 1 & \sqrt{\frac{P}{EI}} & 0 \\ 0 & 1 & \sqrt{\frac{P}{EI}}\cos\left(\sqrt{\frac{P}{EI}}l\right) & -\sqrt{\frac{P}{EI}}\sin\left(\sqrt{\frac{P}{EI}}l\right) \end{pmatrix} . \tag{8.94}$$

Nontrivial solution exists provided that $\det M = 0$, i.e. when

$$\sqrt{\frac{P}{EI}}l \sin\left(\sqrt{\frac{P}{EI}}l\right) - 2 + 2\cos\left(\sqrt{\frac{P}{EI}}l\right) = 0 . \tag{8.95}$$

This condition can be expressed as

$$\mathcal{D}\left(\frac{1}{2\pi}\sqrt{\frac{P}{EI}}l\right) = 0 ,$$

where the function $\mathcal{D}(p)$ is defined by

$$\mathcal{D}(p) = 2\pi p \sin(2\pi p) - 2 + 2\cos(2\pi p) .$$

This equation can be solved graphically, as can be seen in Fig. 8.6, which plots the function $\mathcal{D}(p)$. Note that for the first solution occurs at $p = 1$, thus the first critical load is given by

$$P_1 = \frac{4\pi^2 EI}{l^2} , \tag{8.96}$$

and the corresponding mode shape is given by

$$y_1(x) = A + Bx + C\sin\frac{2\pi x}{l} + D\cos\frac{2\pi x}{l} \ . \tag{8.97}$$

The boundary conditions are satisfied provided that

$$A + D = A + Bl + D = 0 \ , \tag{8.98}$$

$$B + \frac{2\pi}{l}C = B + \frac{2\pi}{l}\left[C\cos\frac{2\pi x}{l} - D\sin\frac{2\pi x}{l}\right] = 0 \ , \tag{8.99}$$

thus $A + D = 0$, $B = C = 0$, and therefore

$$y_1(x) = A\left(1 - \cos\frac{2\pi x}{l}\right) \ . \tag{8.100}$$

The relatively simple analysis that has been used above allows calculating the critical loads P_n, above which the nth mode becomes unstable and buckling occurs. However, in this treatment the post-buckling amplitude cannot be determined. For this one has to consider the more general case where the beam deflection may be large.

8.9 Post Buckling

Consider the case where the deflection $y(x,t)$ has the shape of the first buckling configuration $y_1(x)$ for the case of doubly clamped boundary conditions, i.e. [see Eq. (8.100)]

$$\frac{y}{l} = \mathcal{Y}\left(1 - \cos\frac{2\pi x}{l}\right) \ , \tag{8.101}$$

where $\mathcal{Y}$ denotes dimensionless time dependent amplitude. In order to account for a possible asymmetry, the force per unit length f is allowed to be a nonzero constant. The Lagrangian (8.30) for the present case becomes

$$
\begin{aligned}
\mathcal{L}_0 &= \frac{A_{\mathrm{cs}}\rho l^2 \left(\frac{\mathrm{d}\mathcal{Y}}{\mathrm{d}t}\right)^2}{2} \int_0^l \mathrm{d}x \, \left(1 + \cos\frac{2\pi x}{l}\right)^2 \\
&\quad - \frac{N l^2 \mathcal{Y}^2}{2} \left(\frac{2\pi}{l}\right)^2 \int_0^l \mathrm{d}x \, \sin^2\frac{2\pi x}{l} \\
&\quad - \frac{E I l^2 \mathcal{Y}^2}{2} \left(\frac{2\pi}{l}\right)^4 \int_0^l \mathrm{d}x \, \cos^2\frac{2\pi x}{l} \\
&\quad + f l \mathcal{Y} \int_0^l \mathrm{d}x \, \left(1 + \cos\frac{2\pi x}{l}\right) \\
&\quad - \frac{A_{\mathrm{cs}} E l^3 \mathcal{Y}^4}{8} \left(\frac{2\pi}{l}\right)^2 \left(\int_0^l \mathrm{d}x \, \sin^2\frac{2\pi x}{l}\right)^2 ,
\end{aligned}
\tag{8.102}
$$

or

$$
\begin{aligned}
\mathcal{L}_0 &= \frac{A_{\mathrm{cs}}\rho l^2 \left(\frac{\mathrm{d}\mathcal{Y}}{\mathrm{d}t}\right)^2}{2} \frac{3l}{2} - \frac{N l^2 \mathcal{Y}^2}{2} \left(\frac{2\pi}{l}\right)^2 \frac{l}{2} \\
&\quad - \frac{E I l^2 \mathcal{Y}^2}{2} \left(\frac{2\pi}{l}\right)^4 \frac{l}{2} + f l^2 \mathcal{Y} - \frac{\pi^4 A_{\mathrm{cs}} E l^4 \mathcal{Y}^4}{2 l^3} ,
\end{aligned}
\tag{8.103}
$$

thus

$$
\mathcal{L}_0 = T_0 - U_0 , \tag{8.104}
$$

where the kinetic energy T_0 is given by

$$
T_0 = \frac{m_0 l^2 \left(\frac{\mathrm{d}\mathcal{Y}}{\mathrm{d}t}\right)^2}{2} , \tag{8.105}
$$

where

$$
m_0 = \frac{3 A_{\mathrm{cs}} \rho l}{2} , \tag{8.106}
$$

and where the potential energy U_0 is given by

$$
U_0 = \frac{m_0 \omega_0^2 l^2}{2} \left(\eta_1 \mathcal{Y} + \eta_2 \mathcal{Y}^2 + \eta_4 \mathcal{Y}^4\right) , \tag{8.107}
$$

where

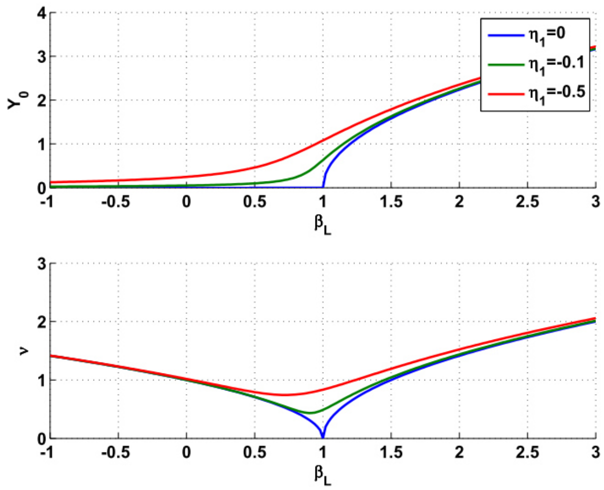


Fig. 8.7. The parameters $\mathcal{Y}_0$ and ν vs. β_{L} for three values of η_1 and for the case where $\eta_4 = 1$.

$$\eta_1 = -\frac{2f}{m_0\omega_0^2} , \tag{8.108}$$

$$\eta_2 = 1 - \beta_{\mathrm{L}} , \tag{8.109}$$

$$\eta_4 = \frac{\pi^4 A_{\mathrm{cs}} E}{m_0\omega_0^2 l} , \tag{8.110}$$

and where

$$\beta_{\mathrm{L}} = -\frac{Nl^2}{4\pi^2 EI} , \tag{8.111}$$

and

$$\omega_0^2 = \frac{(2\pi)^4 E^2 I}{3\rho N l^4} . \tag{8.112}$$

The potential U_0 can be expanded near one of its local minima points $\mathcal{Y}_0$ to second order in $\mathcal{Y} - \mathcal{Y}_0 \equiv \xi$ as

$$U_0 = \frac{m_0\omega_0^2 l^2}{2} \left(u_0 + \nu^2\xi^2 + O\left(\xi^3\right)\right) , \tag{8.113}$$

where u_0, $\mathcal{Y}_0$ and $\nu = \sqrt{\eta_2 + 6\eta_4\mathcal{Y}_0^2}$ are constants. The parameters $\mathcal{Y}_0$ and ν are plotted as a function of β_{L} in Fig. 8.7 for three values of the asymmetry parameter η_1 and for the value 1 of the nonlinear parameter η_4.

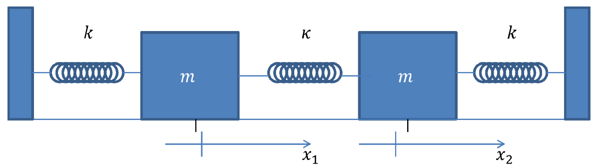


Fig. 8.8. Two coupled resonators.

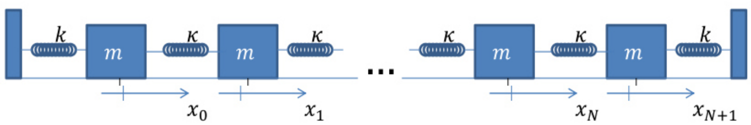


Fig. 8.9. One dimensional chain.

8.10 Problems

1. Find the normal modes of the system that is shown in Fig. 8.8.
2. The system that is shown in Fig. 8.8 is in thermal equilibrium at temperature T. Calculate the expectation value $\langle x_1^2 \rangle$.
3. Consider the system shown in Fig. 8.9, which consists of $N+2$ objects having mass m each and $N+3$ springs. The spring constant of the first (from the left) spring and the last one is k, whereas all other springs have spring constant of κ. To effectively introduce boundary conditions of 'fixed ends' it is assumed that $k \gg \kappa$. Find the normal modes of the system.
4. Consider a system having a Lagrangian $\mathcal{L}$ that is given by
$$\mathcal{L} = \frac{1}{2}\dot{X}M\dot{X}^T - \frac{1}{2}XKX^T \ , \tag{8.114}$$
where $X = (x_1, x_2, \cdots, x_N)$ is a vector of coordinates and where both M and K are $N \times N$ symmetric matrices of constants. Show that the angular frequencies ω of normal modes can be found by solving
$$\det\left(\omega^2 M - K\right) = 0 \ . \tag{8.115}$$
5. The state of a beam-like system is described by the height function $y(x,t)$. The Lagrangian is given by
$$\mathcal{L} = \int_{-l/2}^{l/2} \mathrm{d}x \left[\frac{\Upsilon}{2}\left(\frac{\partial y}{\partial t}\right)^2 - \frac{N}{2}\left(\frac{\partial y}{\partial x}\right)^2 - \frac{\zeta}{12}\left(\frac{\partial y}{\partial x}\right)^4\right] . \tag{8.116}$$
The boundary conditions at the end points $x=0$ and $x=l$ are taken to be given by $y=0$ and $\partial y/\partial x = 0$. Find an equation of motion for $y(x,t)$.

6. A string made of metal having a coefficient of thermal expansion α_1 is attached at both ends to a substrate having coefficient of thermal expansion α_2. The supporting substrate is assumed to be much larger than the suspended string. The mass per unit length of the string is Υ, the cross section area of the string is A and its Young's modulus is E. At reference temperature T_0 the tension in the string is N_0 and the distance between the clamping points at both ends of the string is l_0. Calculate the resonance frequencies f_n of the string at temperature T. For the calculation of f_n assume that Eq. (8.71) holds, i.e. assume that the tension is sufficiently large, and consequently, corrections to f_n due to stiffness are negligibly small.
7. **Stiff string** - Consider a doubly clamped beam having mass density (i.e. mass per unit length) Υ, Young's modulus E and moment of inertia I (corresponding to the plane at which the beam is assume to vibrate). A tension N is applied at both ends of the beam. The distance between the clamping points is l. The dimensionless parameter $\beta = (1/l)\sqrt{EI/N}$ characterized the relative effect of stiffness on the properties of the beam. Calculate the resonance frequencies for the case where $\beta \ll 1$, i.e. for the case where stiffness can be considered as a small perturbation.
8. Consider a beam having Young's modulus E, length l and moment of inertia I. One end of the beam at $x = 0$ is clamped, whereas a force P is applied in the y direction (perpendicularly to the beam axis) to the other end, which is otherwise free. Find the deflection of the end of the beam.
9. Consider a beam having Young's modulus E, mass density per unit length Υ, cross section area A, coefficient of thermal expansion α_1 and moment of inertia I. The beam is doubly clamped to a substrate having coefficient of thermal expansion α_2. The supporting substrate is assumed to be much larger than the suspended beam. At a reference temperature T_0 the tension in the beam is N_0 and the distance between the clamping points at both ends of the beam is l_0. At what temperature T_{c} the beam is expected to buckle?
10. Consider a doubly clamped tension-free beam having Young's modulus E, length l, circular cross section having radius a and mass density per unit volume ρ. The beam is placed in a gravitational field perpendicular to its axis having acceleration constant g . Determine the shape of the beam that is bent by its own wight.
11. Consider a string, which is doubly clamped to a substrate at the points $(x, y) = (0, 0)$ and $(x, y) = (l, 0)$. The motion of the string's axis, which is described by the height function $y(x, t)$, is assumed to be exclusively in the xy plane. The mass density per unit length is Υ and the tension in the string for the straight case where $y = 0$ is N. The string is in thermal equilibrium at temperature T. Let $y_0 = y(l/2, t)$ be the displacement of the string's central point. Calculate the expectation value $\langle y_0^2 \rangle$.

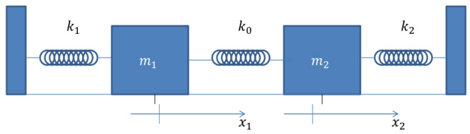


Fig. 8.10.

12. Find the angular frequencies ω of the normal modes of the system that is shown in Fig. 8.10. Assume that

$$\begin{aligned} m_1 &= m \ , \\ m_2 &= 2m \ , \\ k_1 &= 4k \ , \\ k_2 &= 8k \ , \\ k_0 &= k \ . \end{aligned}$$

13. Consider a tension-free beam having Young's modulus E, length l, mass density per unit length Υ and moment of inertia I. One end of the beam at $x = 0$ is pinned, i.e. $y = 0$ and $y'' = 0$ at $x = 0$, whereas the boundary conditions at $x = l$ are assumed to be $y' = 0$ and $y''' = 0$. Find the resonance frequencies f_n of the normal modes.

8.11 Solutions

1. In the absent of the middle spring (having spring constant κ) the system consists of two decoupled resonators having the same angular resonance frequency given by $\sqrt{k/m}$. However, due to the coupling that is introduced by the middle spring the two resonators become influenced by each other. The Lagrangian of the system is given by [see Eq. (1.16)]

$$\mathcal{L} = \frac{m\dot{x}_1^2}{2} + \frac{m\dot{x}_2^2}{2} - \frac{kx_1^2}{2} - \frac{kx_2^2}{2} - \frac{\kappa (x_2 - x_1)^2}{2} \ . \tag{8.117}$$

Using Eq. (1.8) one finds the following Euler-Lagrange coupled equations of motion

$$m\ddot{x}_1 + kx_1 + \kappa (x_1 - x_2) = 0 \ , \tag{8.118}$$

$$m\ddot{x}_2 + kx_2 + \kappa (x_2 - x_1) = 0 \ . \tag{8.119}$$

In matrix form the Lagrangian (8.117) can be expressed as

$$\mathcal{L} = \frac{1}{2} \begin{pmatrix} \dot{x}_1 & \dot{x}_2 \end{pmatrix} M \begin{pmatrix} \dot{x}_1 \\ \dot{x}_2 \end{pmatrix} - \frac{1}{2} \begin{pmatrix} x_1 & x_2 \end{pmatrix} K \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} \ , \tag{8.120}$$

where

$$M = m \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} , \tag{8.121}$$

and where

$$\begin{aligned} K &= \begin{pmatrix} k+\kappa & -\kappa \\ -\kappa & k+\kappa \end{pmatrix} \\ &= (k+\kappa) \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} - \kappa \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} . \end{aligned} \tag{8.122}$$

The fact that the two equations of motion (8.118) and (8.119) are coupled to each other can be attributed to the fact that the matrix K is not diagonal. The matrix K can be diagonalized by the following transformation

$$K = U^{-1} \begin{pmatrix} k & 0 \\ 0 & k+2\kappa \end{pmatrix} U , \tag{8.123}$$

where

$$U = U^{-1} = \begin{pmatrix} \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{2}} \end{pmatrix} . \tag{8.124}$$

This implies that in terms of the coordinates x'_1 and x'_2, which are related to x_1 and x_2 by

$$\begin{pmatrix} x'_1 \\ x'_2 \end{pmatrix} = U \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \frac{1}{\sqrt{2}} \begin{pmatrix} x_1 + x_2 \\ x_1 - x_2 \end{pmatrix} , \tag{8.125}$$

the Lagrangian is given by (note that $M = U^{-1}MU$ and that U is Hermitian)

$$\begin{aligned} \mathcal{L} &= \frac{1}{2} \left(\dot{x}'_1 \ \dot{x}'_2 \right) M \begin{pmatrix} \dot{x}'_1 \\ \dot{x}'_2 \end{pmatrix} \\ &\quad - \frac{1}{2} \left(x'_1 \ x'_2 \right) \begin{pmatrix} k & 0 \\ 0 & k+2\kappa \end{pmatrix} \begin{pmatrix} x'_1 \\ x'_2 \end{pmatrix} . \end{aligned} \tag{8.126}$$

The resultant Euler-Lagrange equations of motion [see Eq. (1.8)] are given by

$$m\ddot{x}'^2_1 + kx'_1 = 0 , \tag{8.127}$$

$$m\ddot{x}'^2_2 + (k+2\kappa)\, x'_2 = 0 . \tag{8.128}$$

These equations describe two decoupled harmonic resonators. The first having the coordinate $x'_1 = 2^{-1/2} (x_1 + x_2)$ has angular resonance frequency $\omega'_1 = \sqrt{k/m}$, whereas the second having the coordinate $x'_2 =$

$2^{-1/2}(x_1 - x_2)$ has angular resonance frequency $\omega_1' = \sqrt{(k+2\kappa)/m}$. These two decoupled harmonic resonators constitute the normal modes of the system.

2. In terms of the coordinates x_1' and x_2', which are related to x_1 and x_2 by

$$x_1' = \frac{x_1 + x_2}{\sqrt{2}} , \tag{8.129}$$

$$x_2' = \frac{x_1 - x_2}{\sqrt{2}} , \tag{8.130}$$

the Lagrangian is given by [see Eq. (8.126)]

$$\mathcal{L} = \frac{m\dot{x}_1'^2}{2} + \frac{m\dot{x}_2'^2}{2} - \frac{kx_1'^2}{2} - \frac{(k+2\kappa)\, x_2'^2}{2} . \tag{8.131}$$

In thermal equilibrium the following holds [see Eq. (4.19)]

$$\left\langle x_1'^2 \right\rangle = \frac{k_{\mathrm{B}}T}{k} , \tag{8.132}$$

$$\left\langle x_2'^2 \right\rangle = \frac{k_{\mathrm{B}}T}{k+2\kappa} , \tag{8.133}$$

and

$$\langle x_1' x_2' \rangle = 0 , \tag{8.134}$$

and thus by using the inverse transformation

$$x_1 = \frac{x_1' + x_2'}{\sqrt{2}} , \tag{8.135}$$

one finds that

$$\begin{aligned} \left\langle x_1^2 \right\rangle &= \frac{\left\langle x_1'^2 + x_2'^2 + 2x_1'x_2' \right\rangle}{2} \\ &= \frac{k_{\mathrm{B}}T}{2k} + \frac{k_{\mathrm{B}}T}{2(k+2\kappa)} \\ &= \frac{k_{\mathrm{B}}T}{k}\left(1 - \frac{\frac{\kappa}{k}}{1+\frac{2\kappa}{k}}\right) . \end{aligned} \tag{8.136}$$

3. The Lagrangian of the system is given by [see Eq. (1.16)]

$$\begin{aligned} \tilde{\mathcal{L}} = & \frac{m\dot{x}_0^2}{2} + \frac{m\dot{x}_1^2}{2} + \cdots + \frac{m\dot{x}_N^2}{2} + \frac{m\dot{x}_{N+1}^2}{2} \\ & - \frac{kx_0^2}{2} + \frac{\kappa (x_1 - x_0)^2}{2} + \frac{\kappa (x_2 - x_1)^2}{2} + \cdots \\ & + \frac{\kappa (x_{N+1} - x_N)^2}{2} + \frac{kx_{N+1}^2}{2} . \end{aligned} \tag{8.137}$$

In the limit where $k/\kappa \to \infty$ the coordinates x_0 and x_{N+1} are assumed to vanish at all time due to the strong confinement of the end springs having spring constant k. As a function of the other coordinates, which are labeled in a vector form as $X = (x_1, x_2, \cdots, x_N)$, the Lagrangian is given in a matrix form by

$$\mathcal{L} = \frac{1}{2}\dot{X}M\dot{X}^T - \frac{1}{2}XKX^T \ , \tag{8.138}$$

where the matrix M is given by $M = mI_N$, where I_N is the $N \times N$ identity matrix, and where the matrix K is given by $K = \kappa\,(2I_N - S)$, where the elements of the $N \times N$ matrix S are given by $S_{nm} = \delta_{|n-m|,1}$, i.e.

$$S = \begin{pmatrix} 0\,1\,0 \cdots 0\,0\,0 \\ 1\,0\,1 \cdots 0\,0\,0 \\ 0\,1\,0 \cdots 0\,0\,0 \\ \vdots\,\vdots\,\vdots \\ 0\,0\,0 \qquad 0\,1\,0 \\ 0\,0\,0 \qquad 1\,0\,1 \\ 0\,0\,0 \qquad 0\,1\,0 \end{pmatrix} . \tag{8.139}$$

To diagonalize S we consider a vector having the following form $V_\theta = (\sin(\theta), \sin(2\theta), \cdots, \sin(N\theta))$. Using the general identity

$$\sin x + \sin y = 2\cos\frac{x-y}{2}\sin\frac{x+y}{2} \ , \tag{8.140}$$

one finds that the following holds

$$SV_\theta^T = 2\cos\theta V_\theta^T + \tilde{V} \ , \tag{8.141}$$

where $\tilde{V} = (0, 0, \cdots, -\sin((N+1)\theta))$. Thus V_θ is an eigenvector of S with an eigenvalue $2\cos\theta$ provided that

$$\sin((N+1)\theta) = 0 \ . \tag{8.142}$$

This condition is satisfied when $\theta = k\pi/(N+1)$, where k is integer. Consider the set of vectors

$$V_k = \left(\frac{2}{N+1}\right)^{1/2}(\sin(\theta_k), \sin(2\theta_k), \cdots, \sin(N\theta_k)) \ , \tag{8.143}$$

where

$$\theta_k = \frac{k\pi}{N+1} \ , \tag{8.144}$$

and where $k \in \{1, 2, \cdots, N\}$. The following holds

$$KV_k^T = \kappa \left(2I_N - S\right) V_k^T = 2\kappa \left(1 - \cos\theta_k\right) V_k^T \ , \tag{8.145}$$

thus

$$KV_k^T = \kappa_k V_k^T \ , \tag{8.146}$$

where

$$\kappa_k = 4\kappa \sin^2 \frac{\theta_k}{2} \ , \tag{8.147}$$

i.e. V_k^T is an eigenvector of K with eigenvalue κ_k. Furthermore, as is shown below, the vectors $\{V_k\}$ are orthonormal, i.e. $V_n V_m^T = \delta_{n,m}$. The following holds

$$V_n V_m^T = \frac{2}{N+1} \sum_{l=1}^{N} \sin\left(l\theta_n\right) \sin\left(l\theta_m\right) \ . \tag{8.148}$$

With the help of the general identity

$$\sin x \sin y = \frac{1}{2} \left[\cos\left(x - y\right) - \cos\left(x + y\right)\right] \ , \tag{8.149}$$

this becomes

$$V_n V_m^T = \frac{1}{N+1} \sum_{l=1}^{N} \left[\cos\left(l\left(\theta_n - \theta_m\right)\right) - \cos\left(l\left(\theta_n + \theta_m\right)\right)\right] \ , \tag{8.150}$$

or

$$V_n V_m^T = \frac{S\left(n - m\right) - S\left(n + m\right)}{N+1} \ , \tag{8.151}$$

where

$$S\left(k\right) = \sum_{l=1}^{N} \cos \frac{lk\pi}{N+1} \ . \tag{8.152}$$

The following holds $S\left(0\right) = N$ since $\cos\left(0\right) = 1$. For the case where k is a nonzero integer one finds by using the identity

$$\sum_{l=1}^{N} q^l = \frac{q\left(q^{N+1} - 1\right)}{q - 1} - q^{N+1} \ , \tag{8.153}$$

that

$$S\left(k\right) = \operatorname{Re} \sum_{l=1}^{N} e^{\frac{ilk\pi}{N+1}} = \operatorname{Re}\left(\frac{e^{ik\pi} - 1}{1 - e^{-\frac{ik\pi}{N+1}}} - e^{ik\pi}\right) \ . \tag{8.154}$$

Thus $S(k) = -1$ for even k, since for this case $e^{ik\pi} = 1$. For odd k, on the other hand, $e^{ik\pi} = -1$, and consequently

$$S(k) = \mathrm{Re}\left(-\frac{1+e^{-\frac{ik\pi}{N+1}}}{1-e^{-\frac{ik\pi}{N+1}}}\right) = \mathrm{Re}\left(i\cot\frac{k\pi}{2(N+1)}\right) = 0 \; . \tag{8.155}$$

Combining all these results one find that

$$V_n V_m^T = \delta_{n,m} \; . \tag{8.156}$$

The unitary matrix U, which is built using the eigenvectors $\{V_n\}$, i.e.

$$U_{n,m} = \left(\frac{2}{N+1}\right)^{1/2} \sin\frac{nm\pi}{N+1} \; , \tag{8.157}$$

allows diagonalization of K, i.e. $U^\dagger K U$ is diagonal with eigenvalues given by Eq. (8.147). The corresponding angular resonance frequencies are given by

$$\omega_k = \sqrt{\frac{\kappa_k}{m}} = 2\sqrt{\frac{\kappa}{m}} \sin\frac{k\pi}{2(N+1)} \; . \tag{8.158}$$

4. Using the fact that both M and K are symmetric one finds that the set of Euler-Lagrange equations can be expressed in a matrix form as [see Eq. (1.8)]

$$M\ddot{X} + KX = 0 \; . \tag{8.159}$$

Seeking a solution having the form

$$X = X_0 e^{-i\omega t} \; , \tag{8.160}$$

where X_0 is time independent, yields the following equation for X_0

$$\left(\omega^2 M - K\right) X_0 = 0 \; . \tag{8.161}$$

The requirement that a nontrivial solution exists leads to

$$\det\left(\omega^2 M - K\right) = 0 \; . \tag{8.162}$$

5. The equation of motion is found using Eq. (8.38)

$$\varUpsilon\frac{\partial^2 y}{\partial t^2} = N\frac{\partial^2 y}{\partial x^2} + \zeta\left(\frac{\partial y}{\partial x}\right)^2 \frac{\partial^2 y}{\partial x^2} \; . \tag{8.163}$$

6. The assumption that the supporting substrate is much larger than the suspended string implies that the thermal expansion of the substrate due to the temperature change $\varDelta T = T - T_0$ is not significantly affected by

the fact that a string is attached to it. Thus, to a good approximation the distance between the clamping points at both ends of the string becomes $l = l_0 (1 + \alpha_2 \Delta T)$ at temperature T. The corresponding added strain is $\alpha_2 \Delta T$. The normal stress σ can be expressed in terms of the tension in the string N as $\sigma = N/A$. The stress-strain relation at the reference temperature T_0 is given by

$$\epsilon_0 = \frac{\sigma_0}{E} , \tag{8.164}$$

where $\sigma_0 = N_0/A$, whereas at temperature T the following holds [see Eq. (7.15)]

$$\epsilon_0 + \alpha_2 \Delta T = \frac{\sigma}{E} + \alpha_1 \Delta T , \tag{8.165}$$

where $\sigma = N/A$, thus

$$N = N_0 + AE (\alpha_2 - \alpha_1) \Delta T . \tag{8.166}$$

With the help of Eq. (8.71) one finds that the resonance frequencies are given by

$$f_n = \sqrt{\frac{N_0}{\Upsilon}} \frac{n}{2l} \sqrt{1 + \frac{AE (\alpha_2 - \alpha_1) \Delta T}{N_0}} . \tag{8.167}$$

7. The equation of motion is given by

$$\Upsilon \ddot{y} = N y'' - EI y'''' , \tag{8.168}$$

The boundary conditions at the clamped points, which are taken to be located at $x = \pm l/2$, are given by

$$y \left(\pm \frac{l}{2} \right) = 0 , \tag{8.169}$$

and

$$y' \left(\pm \frac{l}{2} \right) = 0 . \tag{8.170}$$

Substituting a solution having the form

$$y (x, t) = \exp \left(\frac{\mu x}{l} - i \omega t \right) \tag{8.171}$$

yields

$$(-i\omega)^2 \Upsilon = N \left(\frac{\mu}{l} \right)^2 - EI \left(\frac{\mu}{l} \right)^4 , \tag{8.172}$$

or

$$-\beta^2\mu^4 + \mu^2 + \gamma^2 = 0 \; , \tag{8.173}$$

where

$$\beta = \frac{1}{l}\sqrt{\frac{EI}{N}} \; , \tag{8.174}$$

and

$$\gamma = \sqrt{\frac{\Upsilon}{N}} l\omega \; . \tag{8.175}$$

The solutions are μ_1, $i\mu_2$, $-\mu_1$ and $-i\mu_2$ where

$$\mu_1 = \frac{1}{\sqrt{2}\beta}\sqrt{\sqrt{1+4\beta^2\gamma^2}+1} \; , \tag{8.176}$$

$$\mu_2 = \frac{1}{\sqrt{2}\beta}\sqrt{\sqrt{1+4\beta^2\gamma^2}-1} \; . \tag{8.177}$$

Note that the following holds

$$\mu_1 = \frac{1}{\beta}\sqrt{1+\beta^2\mu_2^2} \; , \tag{8.178}$$

and

$$\gamma^2 = \beta^2\mu_1^2\mu_2^2 = \mu_2^2\left(1+\beta^2\mu_2^2\right) \; . \tag{8.179}$$

The boundary conditions for an even solution having the form

$$y = \left(A\cosh\frac{\mu_1 x}{l} + B\cos\frac{\mu_2 x}{l}\right)\exp\left(-i\omega t\right) \tag{8.180}$$

can be written as

$$\begin{pmatrix} \cosh\frac{\mu_1}{2} & \cos\frac{\mu_2}{2} \\ \mu_1\sinh\frac{\mu_1}{2} & -\mu_2\sin\frac{\mu_2}{2} \end{pmatrix}\begin{pmatrix} A \\ B \end{pmatrix} = 0 \; . \tag{8.181}$$

The condition for the existence of a nontrivial solution is given by

$$-\cosh\frac{\mu_1}{2}\mu_2\sin\frac{\mu_2}{2} - \cos\frac{\mu_2}{2}\mu_1\sinh\frac{\mu_1}{2} = 0 \; , \tag{8.182}$$

or by [see Eq. (8.178)]

$$\mu_2\tan\frac{\mu_2}{2} = -\frac{1}{\beta}\sqrt{1+\beta^2\mu_2^2}\tanh\frac{\sqrt{1+\beta^2\mu_2^2}}{2\beta} \; , \tag{8.183}$$

thus

$$\frac{\cot\frac{\mu_2}{2}}{\mu_2} = -\frac{\beta\coth\frac{\sqrt{1+\beta^2\mu_2^2}}{2\beta}}{\sqrt{1+\beta^2\mu_2^2}} \ . \tag{8.184}$$

Similarly, the boundary conditions for an odd solution having the form

$$y\left(x\right) = \left(A\sinh\frac{\mu_1 x}{l} + B\sin\frac{\mu_2 x}{l}\right)\exp\left(-i\omega t\right) \tag{8.185}$$

can be written as

$$\begin{pmatrix} \sinh\frac{\mu_1}{2} & \sin\frac{\mu_2}{2} \\ \mu_1\cosh\frac{\mu_1}{2} & \mu_2\cos\frac{\mu_2}{2} \end{pmatrix}\begin{pmatrix} A \\ B \end{pmatrix} = 0 \ . \tag{8.186}$$

The condition for the existence of a nontrivial solution is given by

$$\sinh\frac{\mu_1}{2}\mu_2\cos\frac{\mu_2}{2} - \sin\frac{\mu_2}{2}\mu_1\cosh\frac{\mu_1}{2} = 0 \ , \tag{8.187}$$

or by [see Eq. (8.178)]

$$\mu_2\cot\frac{\mu_2}{2} = \frac{1}{\beta}\sqrt{1+\beta^2\mu_2^2}\coth\frac{\sqrt{1+\beta^2\mu_2^2}}{2\beta} \ , \tag{8.188}$$

thus

$$\frac{\tan\frac{\mu_2}{2}}{\mu_2} = \frac{\beta\tanh\frac{\sqrt{1+\beta^2\mu_2^2}}{2\beta}}{\sqrt{1+\beta^2\mu_2^2}} \ . \tag{8.189}$$

To third order in the small parameter β Eq. (8.184) is expanded as

$$\frac{\cot\frac{\mu_2}{2}}{\mu_2} = -\beta\left(1 - \frac{\mu_2^2\beta^2}{2}\right) + O\left(\beta^5\right) \ . \tag{8.190}$$

Taking

$$\mu_2^{(n)} = \pi n + \delta_n \ , \tag{8.191}$$

where

$$n = 1, 3, 5, \cdots \ , \tag{8.192}$$

and using the expansion

$$\delta_n = a_{1n}\beta + a_{2n}\beta^2 + O\left(\beta^3\right) \ , \tag{8.193}$$

lead to

$$-\frac{\delta_n}{2\pi n}\left(1-\frac{\delta_n}{\pi n}\right)+O\left(\delta_n^3\right)=-\beta\left(1-\frac{\mu_2^2\beta^2}{2}\right)+O\left(\beta^5\right)\ ,$$

thus

$$a_{1n}=2\pi n\ , \tag{8.194}$$
$$a_{2n}=4\pi n\ , \tag{8.195}$$

therefore

$$\mu_2^{(n)}=\pi n\left(1+2\beta+4\beta^2\right)+O\left(\beta^3\right)\ . \tag{8.196}$$

Similarly, to third order in β Eq. (8.189) is expanded as

$$\frac{\tan\frac{\mu_2}{2}}{\mu_2}=\beta\left(1-\frac{\mu_2^2\beta^2}{2}\right)+O\left(\beta^5\right)\ . \tag{8.197}$$

Taking

$$\mu_2^{(n)}=\pi n+\delta_n\ , \tag{8.198}$$

where

$$n=2,4,6,\cdots\ , \tag{8.199}$$

and using the expansion

$$\delta_n=a_{1n}\beta+a_{2n}\beta^2+O\left(\beta^3\right)\ , \tag{8.200}$$

lead to

$$\frac{\delta_n}{2\pi n}\left(1-\frac{\delta_n}{\pi n}\right)+O\left(\delta_n^3\right)=\beta\left(1-\frac{\mu_2^2\beta^2}{2}\right)+O\left(\beta^5\right)\ ,$$

thus

$$a_{1n}=2\pi n\ , \tag{8.201}$$
$$a_{2n}=4\pi n\ , \tag{8.202}$$

therefore

$$\mu_2^{(n)}=\pi n\left(1+2\beta+4\beta^2\right)+O\left(\beta^3\right)\ . \tag{8.203}$$

The result for the odd case (8.189), which is identical to the result for the even case (8.184), together with Eq. (8.179) yield

$$\begin{aligned}\gamma&=\mu_2\sqrt{1+\beta^2\mu_2^2}\\&=\pi n\left(1+2\beta+\left(\frac{\pi^2n^2}{2}+4\right)\beta^2\right)+O\left(\beta^3\right)\ .\end{aligned} \tag{8.204}$$

The eigen frequencies $f_n = \omega_n / 2\pi$ are thus given by

$$\sqrt{\frac{N}{\Upsilon}} \frac{\gamma}{2l\pi} = f$$

$$f_n = \frac{n}{2l}\sqrt{\frac{N}{\Upsilon}}\left(1 + 2\beta + \left(\frac{\pi^2 n^2}{2} + 4\right)\beta^2\right) + O\left(\beta^3\right) \ , \tag{8.205}$$

or

$$f_n = \frac{n}{2l}\sqrt{\frac{N}{\Upsilon}}\left(1 + \frac{2}{l}\sqrt{\frac{EI}{N}} + \left(\frac{\pi^2 n^2}{2} + 4\right)\frac{EI}{Nl^2}\right) + O\left(\beta^3\right) \ . \tag{8.206}$$

8. For the static (i.e. time independent) case the height function satisfies the following equation

$$y'''' = \frac{P}{EI}\delta\left(x - l\right) \ , \tag{8.207}$$

where $\delta\left(\right)$ is the delta function. The boundary conditions at the clamped end at $x = 0$ are $y = 0$ and $y' = 0$. By integrating Eq. (8.207) from $x = l - \varepsilon$ to $x = l + \varepsilon$ and taking the limit $\varepsilon \rightarrow 0$ one finds that $y''' = -P/EI$ at $x = l$. In addition in the free end at $x = l$ the boundary condition $y'' = 0$ must be satisfied. Integrating the equation $y'''' = 0$ four times from 0 to x yields

$$\begin{aligned} y''' &= A \ , \\ y'' &= Ax + B \ , \\ y' &= A\frac{x^2}{2} + Bx + C \ , \\ y &= A\frac{x^3}{6} + B\frac{x^2}{2} + Cx + D \ , \end{aligned}$$

where the constants A, B, C and D are determined by the boundary conditions. The solution is easily found to be

$$y\left(x\right) = \frac{P}{6EI}\left(3lx^2 - x^3\right) \ , \tag{8.208}$$

and thus the deflection at the end of the beam is

$$y\left(l\right) = \frac{Pl^3}{3EI} \ . \tag{8.209}$$

9. The tension at temperature T is given by [see Eq. (8.166)]

$$N = N_0 + AE\left(\alpha_2 - \alpha_1\right)\left(T - T_0\right) \ . \tag{8.210}$$

Buckling occurs when $N = -P_1$, where $P_1 = 4\pi^2 EI/l_0^2$ is the critical load for a doubly clamped beam [see Eq. (8.96)], and thus

$$T_{\mathrm{c}} = T_0 - \frac{N_0 + \frac{4\pi^2 EI}{l_0^2}}{AE\left(\alpha_2 - \alpha_1\right)} \ . \tag{8.211}$$

10. The moment of inertia I is given by Eq. (8.5)

$$I = \frac{\pi a^4}{4} \ . \tag{8.212}$$

The height function $y(x)$ satisfies [see Eq. (8.72)]

$$\frac{\partial^4 y}{\partial x^4} = \frac{4\rho g}{Ea^2}. \tag{8.213}$$

The boundary conditions are $y(\pm l/2) = 0$ and $y'(\pm l/2) = 0$. Consider a solution having the form

$$y(x) = A + Bx^2 + \frac{\rho g}{6Ea^2}x^4 \ , \tag{8.214}$$

where A and B are constants, which are determined by the boundary conditions

$$A = \frac{\rho g l^4}{96Ea^2} \ , \tag{8.215}$$

$$B = -\frac{\rho g l^2}{12Ea^2} \ , \tag{8.216}$$

thus

$$y(x) = \frac{\rho g l^4}{Ea^2}\left[\frac{1}{96} - \frac{1}{12}\left(\frac{x}{l}\right)^2 + \frac{1}{6}\left(\frac{x}{l}\right)^4\right] \ . \tag{8.217}$$

11. The general solution $y(x,t)$ can be expanded as

$$y(x,t) = \sum_{n=1}^{\infty} c_n(t)\,\mathcal{Y}_n(x) \ . \tag{8.218}$$

where [see Eq. (8.59)]

$$\mathcal{Y}_n(x) = \sqrt{\frac{2}{l}}\sin\frac{\pi n x}{l} \ , \tag{8.219}$$

and where

$$k_n = \frac{\pi n}{l} \ . \tag{8.220}$$

The Lagrangian of the system [see Eq. (8.68)] is given by

$$\mathcal{L} = \Upsilon \sum_{n=1}^{\infty}\left(\frac{\dot{c}_n^2}{2} - \omega_n^2\frac{c_n^2}{2}\right) \ , \tag{8.221}$$

where [see Eq. (8.70)]

$$\omega_n = \sqrt{\frac{N}{\Upsilon}}\frac{\pi n}{l} \ . \tag{8.222}$$

According to the equipartition theorem [see Eq. (4.19)] the following holds in thermal equilibrium

$$\Upsilon \omega_n^2 \left\langle c_n^2 \right\rangle = k_{\mathrm{B}} T \ . \tag{8.223}$$

Moreover, as can be seen from Eq. (4.17) for $n \neq m$ the following holds $\langle c_n c_m \rangle = \langle c_n \rangle \langle c_m \rangle = 0$, thus

$$\langle c_n c_m \rangle = \frac{k_{\mathrm{B}} T}{\Upsilon \omega_n^2} \delta_{n,m} \ . \tag{8.224}$$

Using these results one finds that

$$\begin{aligned}
\left\langle y_0^2 \right\rangle &= \sum_{n,m=1}^{\infty} \mathcal{Y}_n\left(\frac{l}{2}\right) \mathcal{Y}_m\left(\frac{l}{2}\right) \langle c_n c_m \rangle \\
&= \sum_{n=1}^{\infty} \mathcal{Y}_n^2\left(\frac{l}{2}\right) \frac{k_{\mathrm{B}} T}{\Upsilon \omega_n^2} \\
&= \frac{2 k_{\mathrm{B}} T l}{\pi^2 N} \sum_{n=1}^{\infty} \frac{\sin^2 \frac{\pi n}{2}}{n^2} \\
&= \frac{2 l k_{\mathrm{B}} T}{\pi^2 N} \sum_{n=1}^{\infty} \frac{1}{(2n-1)^2} \ .
\end{aligned} \tag{8.225}$$

The identities

$$\sum_{n=1}^{\infty} \frac{1}{(2n-1)^2} = \sum_{n=1}^{\infty} \frac{1}{n^2} - \sum_{n=1}^{\infty} \frac{1}{(2n)^2} = \frac{3}{4} \sum_{n=1}^{\infty} \frac{1}{n^2} \ , \tag{8.226}$$

and

$$\sum_{n=1}^{\infty} \frac{1}{n^2} = \frac{\pi^2}{6} \ , \tag{8.227}$$

lead to

$$\left\langle y_0^2 \right\rangle = \frac{l k_{\mathrm{B}} T}{4N} \ . \tag{8.228}$$

12. The Lagrangian $\mathcal{L}$ of the system is given by [see Eq. 8.117)]

$$\mathcal{L} = \frac{m_1 \dot{x}_1^2}{2} + \frac{m_2 \dot{x}_2^2}{2} - \frac{k_1 x_1^2}{2} - \frac{k_2 x_2^2}{2} - \frac{k_0 (x_2 - x_1)^2}{2} \ . \tag{8.229}$$

In matrix form it can be expressed as

$$\mathcal{L} = \frac{1}{2}\dot{X}M\dot{X}^T - \frac{1}{2}XKX^T \ , \tag{8.230}$$

where

$$M = \begin{pmatrix} m_1 & 0 \\ 0 & m_2 \end{pmatrix} \ , \tag{8.231}$$

$$K = \begin{pmatrix} k_1 + k_0 & -k_0 \\ -k_0 & k_2 + k_0 \end{pmatrix} \ , \tag{8.232}$$

and where $X = (x_1, x_2)$. The angular frequencies ω are found by solving [see Eq. (8.115)]

$$0 = \det\left(\omega^2 M - K\right) = \det\begin{pmatrix} \omega^2 m_1 - k_1 - k_0 & k_0 \\ k_0 & \omega^2 m_2 - k_2 - k_0 \end{pmatrix} \ . \tag{8.233}$$

For the given values of the parameters this becomes

$$0 = \det\begin{pmatrix} \lambda - 5 & 1 \\ 1 & 2\lambda - 9 \end{pmatrix} \ , \tag{8.234}$$

where

$$\lambda = \frac{\omega^2 m}{k} \ . \tag{8.235}$$

The solutions are $\omega = 4\sqrt{k/m}$ and $\omega = (11/2)\sqrt{k/m}$.

13. Substituting a solution having the form

$$y(x,t) = \exp(kx - i\omega t) \tag{8.236}$$

into [see Eq. (8.41)]

$$\Upsilon\frac{\partial^2 y}{\partial t^2} = -EI\frac{\partial^4 y}{\partial x^4} \tag{8.237}$$

yields

$$\omega^2\Upsilon = EIk^4 \ , \tag{8.238}$$

thus the four possible solutions are $k = \mathcal{K}$, $-\mathcal{K}$, $i\mathcal{K}$ and $-i\mathcal{K}$, where

$$\mathcal{K} = \left(\frac{\omega^2\Upsilon}{EI}\right)^{1/4} \ , \tag{8.239}$$

and thus the general solution can be expressed as

$$y(x,t) = e^{-i\omega t}\left[A\cosh(\mathcal{K}x) + B\sinh(\mathcal{K}x) + C\cos(\mathcal{K}x) + D\sin(\mathcal{K}x)\right] \ . \tag{8.240}$$

While the boundary condition $y = 0$ at the pinned end at $x = 0$ yields the condition $A + C = 0$, the other boundary condition at that point $y'' = 0$ yields $A - C = 0$, thus $A = C = 0$. The boundary conditions at $x = l$ yield

$$\begin{pmatrix} \cosh(\mathcal{K}x) & \cos(\mathcal{K}x) \\ \cosh(\mathcal{K}x) & -\cos(\mathcal{K}x) \end{pmatrix} \begin{pmatrix} B \\ D \end{pmatrix} = 0 \; . \tag{8.241}$$

Nontrivial solution exists provided that the determinant of the matrix above vanishes, i.e. provided that

$$\cosh(\mathcal{K}l)\cos(\mathcal{K}l) = 0 \; . \tag{8.242}$$

Since $\cosh(\mathcal{K}l) \neq 0$ provided that $\mathcal{K}$ is real the condition can be written as $\cos(\mathcal{K}l) = 0$, or alternatively as

$$\cos\left(\left(\frac{\omega^2 \Upsilon}{EI}\right)^{1/4} l\right) = 0 \; . \tag{8.243}$$

The solutions for the angular frequencies ω_n are given by

$$\left(\frac{\omega_n^2 \Upsilon}{EI}\right)^{1/4} l = \pi\left(n + \frac{1}{2}\right) \; , \tag{8.244}$$

where n is integer. The corresponding resonance frequencies $f_n = \omega_n/2\pi$ are given by

$$f_n = \frac{\pi\left(n + \frac{1}{2}\right)^2}{2} \frac{1}{l^2} \sqrt{\frac{EI}{\Upsilon}} \; . \tag{8.245}$$

9. Back-Reaction Effects

Displacement detection of a mechanical resonator can be implemented by coupling the resonator to some ancilla system, which is typically externally driven. Commonly, the mutual coupling between the subsystems (mechanical resonator and the ancila system) makes the dynamics of the ancila system dependent on the displacement of the mechanical resonator. Such coupling can be exploited for displacement detection of the mechanical resonator, which can be performed by monitoring the response of the driven ancila system to the externally applied drive. On the other hand, the same coupling unavoidably gives rise to back reaction effects acting back on the mechanical resonator and modifying its properties and its dynamics. Such back reaction effects are first demonstrated in this chapter for the case where a mechanical resonator is coupled to an electromagnetic cavity, forming a so-called optomechanical cavity. The second example deals with bolometric optomechanical coupling, and in third one the case where a mechanical resonator is coupled to a spin system is considered.

9.1 Optomechanical Cavity

The system is schematically depicted in Fig. 9.1. The mass m can move along the x axis in one dimension. A spring having a spring constant k is attached to the mass. The other side of the spring is harnessed to a fixed point on a wall. Let $C(x)$ be the displacement dependent capacitance between the mass and a fixed electrode. It is assumed that this capacitance can be calculated using the parallel plates capacitance formula

$$C(x) = \frac{\varepsilon_0 A}{d_0 - x} , \tag{9.1}$$

where ε_0 is the permittivity constant and A is the effective area. In addition, a source having voltage V and an inductor having inductance L are serially connected between the mass and the fixed electrode.

9.1.1 Equations of Motion

The state of the system is described using the mechanical displacement coordinate x and the coordinate q, which represents the charge stored by the

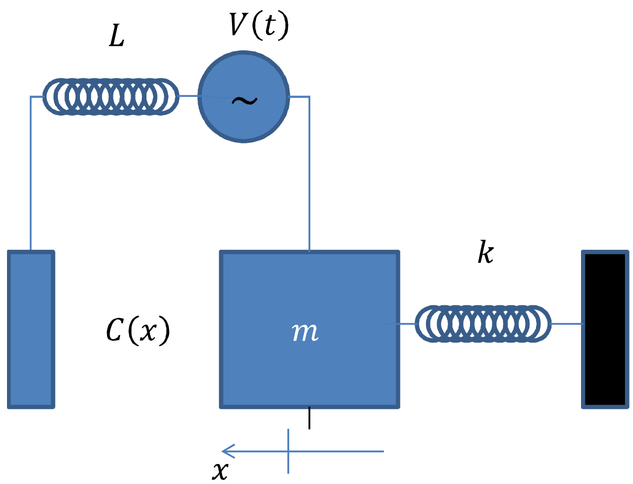


Fig. 9.1. An optomechanical cavity is formed by coupling a mechanical resonator to an LC cavity.

capacitor. The kinetic energy of the mechanical element is $m\dot{x}^2/2$, the potential energy of the mechanical element is $kx^2/2$, the potential energy due to the capacitor is $q^2/2C$, the kinetic energy due to the inductor is $L\dot{q}^2/2$ and the potential energy of the source is given by Vq [see Eq. (2.2)]. Thus the Lagrangian of the system is given by

$$\mathcal{L} = T - U = \frac{m\dot{x}^2}{2} + \frac{L\dot{q}^2}{2} - \frac{kx^2}{2} - \frac{q^2}{2C} - qV \ . \tag{9.2}$$

Using the relation (9.1) and the notation

$$C_0 \equiv C(0) = \frac{\varepsilon_0 A}{d_0} \ , \tag{9.3}$$

the Lagrangian becomes

$$\mathcal{L} = \frac{m\left(\dot{x}^2 - \omega_{\mathrm{m}}^2 x^2\right)}{2} + \frac{L\left(\dot{q}^2 - \omega_{\mathrm{e}}^2 q^2\right)}{2} + \frac{q^2 x}{2C_0 d_0} - qV \ , \tag{9.4}$$

where the angular frequencies ω_{m} and ω_{e} are given by

$$\omega_{\mathrm{m}} = \sqrt{\frac{k}{m}} \ , \tag{9.5}$$

$$\omega_{\mathrm{e}} = \frac{1}{\sqrt{LC_0}} \ . \tag{9.6}$$

The Euler-Lagrange equations for the system are given by [see Eq. (1.8)]

$$\ddot{x} + \omega_{\mathrm{m}}^2 x = \frac{q^2}{2md_0C_0} , \tag{9.7}$$

$$\ddot{q} + \omega_{\mathrm{e}}^2 q = \omega_{\mathrm{e}}^2 q \frac{x}{d_0} - \frac{V}{L} . \tag{9.8}$$

The term on the right hand side of Eq. (9.7) represents the capacitive force acting on the mechanical resonator, whereas the first term on the right hand side of Eq. (9.8) represents the shift in the resonance frequency of the LC cavity due to displacement of the mechanical resonator.

9.1.2 Driving and Damping

Consider the case where the LC circuit is monochromatically driven by applying a voltage V given by

$$V = V_1 \cos(\omega_{\mathrm{p}} t) , \tag{9.9}$$

where both the amplitude V_1 and the angular frequency ω_{p}, which can be expressed as

$$\omega_{\mathrm{p}} = \omega_{\mathrm{e}} + \varDelta , \tag{9.10}$$

are real constants. The detuning $\varDelta$ is assumed to be small in comparison with the angular resonance frequency ω_{e}, i.e. $|\varDelta| \ll \omega_{\mathrm{e}}$. Furthermore, it is assumed that $\omega_{\mathrm{m}} \ll \omega_{\mathrm{e}}$.

It is convenient to introduce the complex variables A_{m} for the mechanical resonator and A_{e} for the LC cavity

$$A_{\mathrm{m}} = \sqrt{\frac{m\omega_{\mathrm{m}}^2}{2}} x + i\sqrt{\frac{m}{2}} \dot{x} , \tag{9.11}$$

$$A_{\mathrm{e}} = \left(\sqrt{\frac{L\omega_{\mathrm{e}}^2}{2}} q + i\sqrt{\frac{L}{2}} \dot{q} \right) e^{i\omega_{\mathrm{p}} t} . \tag{9.12}$$

As can be seen from Eq. (9.12), a frame rotating at angular frequency ω_{p} is employed for describing the dynamics of the LC cavity in term of the complex amplitude A_{e}. The equations of motion (9.7) and (9.8) can be rewritten in terms of the complex amplitudes A_{m} and A_{e} as

$$\dot{A}_{\mathrm{m}} + i\omega_{\mathrm{m}} A_{\mathrm{m}} = i\mathcal{G}\omega_{\mathrm{m}} |A_{\mathrm{e}}|^2 , \tag{9.13}$$

and

$$\dot{A}_{\mathrm{e}} - i\varDelta A_{\mathrm{e}} = i\mathcal{G}\omega_{\mathrm{e}} (A_{\mathrm{m}} + A_{\mathrm{m}}^*) A_{\mathrm{e}} - ib_{\mathrm{p}} , \tag{9.14}$$

where

$$\mathcal{G} = \frac{1}{\sqrt{2kd_0^2}} , \tag{9.15}$$

and

$$b_{\mathrm{p}} = \frac{V_1}{\sqrt{2L}} . \tag{9.16}$$

Note that in Eq. (9.13) above the RWA has been implemented and terms oscillating at angular frequency $2\omega_{\mathrm{p}}$ have been disregarded.

To account for damping in the mechanical resonator the term $\gamma_{\mathrm{m}} A_{\mathrm{m}}$ is added to Eq. (9.13) [see Eq. (3.41)], which can be rewritten as

$$\dot{A}_{\mathrm{m}} + \Theta_{\mathrm{m}} = F_{\mathrm{m}} , \tag{9.17}$$

where

$$\begin{aligned} \Theta_{\mathrm{m}} &= \Theta_{\mathrm{m}} \left(A_{\mathrm{m}}, A_{\mathrm{m}}^*, A_{\mathrm{e}}, A_{\mathrm{e}}^* \right) \\ &= \left(i\omega_{\mathrm{m}} + \gamma_{\mathrm{m}} \right) A_{\mathrm{m}} - i\mathcal{G}\omega_{\mathrm{m}} \left| A_{\mathrm{e}} \right|^2 , \end{aligned} \tag{9.18}$$

and the term $\gamma_{\mathrm{e}} A_{\mathrm{e}}$ (representing an added serial resistor having resistance $2\gamma_{\mathrm{e}} L$) to Eq. (9.14), which becomes

$$\dot{A}_{\mathrm{e}} + \Theta_{\mathrm{e}} = F_{\mathrm{e}} . \tag{9.19}$$

where

$$\begin{aligned} \Theta_{\mathrm{e}} &= \Theta_{\mathrm{e}} \left(A_{\mathrm{m}}, A_{\mathrm{m}}^*, A_{\mathrm{e}}, A_{\mathrm{e}}^* \right) \\ &= \left[-i\Delta - i\mathcal{G}\omega_{\mathrm{e}} \left(A_{\mathrm{m}} + A_{\mathrm{m}}^* \right) + \gamma_{\mathrm{e}} \right] A_{\mathrm{e}} + ib_{\mathrm{p}} . \end{aligned} \tag{9.20}$$

In addition, the stochastic noise terms F_{m} and F_{e} were added to Eqs. (9.17) and (9.19) respectively [see Eq. (5.52)].

9.1.3 Fixed Points

Fixed points are found by solving

$$\Theta_{\mathrm{m}} \left(B_{\mathrm{m}}, B_{\mathrm{m}}^*, B_{\mathrm{e}}, B_{\mathrm{e}}^* \right) = \Theta_{\mathrm{e}} \left(B_{\mathrm{m}}, B_{\mathrm{m}}^*, B_{\mathrm{e}}, B_{\mathrm{e}}^* \right) = 0 . \tag{9.21}$$

With the help of Eqs. (9.18) and (9.20) one finds that

$$\frac{\left| b_{\mathrm{p}} \right|^2}{\left(-\Delta + K E_{\mathrm{e}} \right)^2 + \gamma_{\mathrm{e}}^2} = E_{\mathrm{e}} , \tag{9.22}$$

where

$$E_{\mathrm{e}} = \left| B_{\mathrm{e}} \right|^2 , \tag{9.23}$$

and where

$$K = -\frac{2\mathcal{G}^2\omega_{\mathrm{e}}}{1+\left(\frac{\gamma_{\mathrm{m}}}{\omega_{\mathrm{m}}}\right)^2} \; . \tag{9.24}$$

As can be seen from Eq. (9.22) above [compare with Eq. (6.53)], the coupling with the mechanical resonator introduces Duffing nonlinearity in the response of the LC cavity. Finding E_{e} by solving Eq. (9.22) allows calculating B_{e} using the relation

$$B_{\mathrm{e}} = \frac{ib_{\mathrm{p}}}{i\Delta - \gamma_{\mathrm{e}} - iKE_{\mathrm{e}}} \; , \tag{9.25}$$

and B_{m} using the relation

$$B_{\mathrm{m}} = \frac{\mathcal{G}E_{\mathrm{e}}}{1-i\frac{\gamma_{\mathrm{m}}}{\omega_{\mathrm{m}}}} \; . \tag{9.26}$$

9.1.4 Linearization

To study fluctuation near the fixed points the solution is expressed as

$$A_{\mathrm{m}} = B_{\mathrm{m}} + c_{\mathrm{m}} \; , \tag{9.27a}$$

$$A_{\mathrm{e}} = B_{\mathrm{e}} + c_{\mathrm{e}} \; , \tag{9.27b}$$

where both c_{m} and c_{e} are considered to be small. The linearized equations of motion can be written in a matrix form as [see Eqs. (9.17) and (9.19)]

$$\frac{\mathrm{d}}{\mathrm{d}t}\begin{pmatrix} c_{\mathrm{m}} \\ c_{\mathrm{m}}^* \\ c_{\mathrm{e}} \\ c_{\mathrm{e}}^* \end{pmatrix} + J\begin{pmatrix} c_{\mathrm{m}} \\ c_{\mathrm{m}}^* \\ c_{\mathrm{e}} \\ c_{\mathrm{e}}^* \end{pmatrix} = \begin{pmatrix} F_{\mathrm{m}} \\ F_{\mathrm{m}}^* \\ F_{\mathrm{e}} \\ F_{\mathrm{e}}^* \end{pmatrix} \; , \tag{9.28}$$

where $J = \partial\left(\Theta_{\mathrm{m}}, \Theta_{\mathrm{m}}^*, \Theta_{\mathrm{e}}, \Theta_{\mathrm{e}}^*\right)/\partial\left(A_{\mathrm{m}}, A_{\mathrm{m}}^*, A_{\mathrm{e}}, A_{\mathrm{e}}^*\right)$ is the Jacobian matrix [see Eqs. (9.18) and (9.20)], which at the fixed point $(B_{\mathrm{m}}, B_{\mathrm{m}}^*, B_{\mathrm{e}}, B_{\mathrm{e}}^*)$ can be expressed as

$$J = J_0 + \mathcal{G}V \; . \tag{9.29}$$

The matrix J_0, which represents the response of the system when coupling between the mechanical resonator and the LC cavity is disregarded, can be written in a block form as

$$J_0 = \left(\begin{array}{cc|c} \lambda_{\mathrm{m}} & 0 & \multirow{2}{*}{0} \\ 0 & \lambda_{\mathrm{m}}^* & \\ \hline \multicolumn{2}{c|}{0} & J_{\mathrm{a}} \end{array}\right) \; , \tag{9.30}$$

where J_{a}, which is given by

$$J_{\mathrm{a}} = \begin{pmatrix} \lambda_{\mathrm{e}} & 0 \\ 0 & \lambda_{\mathrm{e}}^* \end{pmatrix} , \tag{9.31}$$

is the Jacobian of the decoupled LC cavity (the ancilla system in the current case), and where the eigenvalues (of the decoupled system) λ_{m} and λ_{e} are given by

$$\lambda_{\mathrm{m}} = i\omega_{\mathrm{m}} + \gamma_{\mathrm{m}} , \tag{9.32}$$

$$\lambda_{\mathrm{e}} = -i\Delta + \gamma_{\mathrm{e}} . \tag{9.33}$$

The contribution of the coupling to the linearized response is described by the matrix V, which is given by

$$V = \begin{pmatrix} 0 & 0 & -i\omega_{\mathrm{m}} B_{\mathrm{e}}^* & -i\omega_{\mathrm{m}} B_{\mathrm{e}} \\ 0 & 0 & i\omega_{\mathrm{m}} B_{\mathrm{e}}^* & i\omega_{\mathrm{m}} B_{\mathrm{e}} \\ -i\omega_{\mathrm{e}} B_{\mathrm{e}} & -i\omega_{\mathrm{e}} B_{\mathrm{e}} & -i\omega_{\mathrm{e}} \left(B_{\mathrm{m}} + B_{\mathrm{m}}^*\right) & 0 \\ i\omega_{\mathrm{e}} B_{\mathrm{e}}^* & i\omega_{\mathrm{e}} B_{\mathrm{e}}^* & 0 & i\omega_{\mathrm{e}} \left(B_{\mathrm{m}} + B_{\mathrm{m}}^*\right) \end{pmatrix} . \tag{9.34}$$

9.1.5 Susceptibility

In general, the Fourier transform of a time dependent variable $O(t)$ is denoted as $O(\omega)$

$$O(t) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} \mathrm{d}\omega \; O(\omega) \, e^{-i\omega t} . \tag{9.35}$$

Applying the Fourier transform to Eq. (9.28) yields

$$(J - i\omega) \begin{pmatrix} c_{\mathrm{m}}(\omega) \\ c_{\mathrm{m}}^*(-\omega) \\ c_{\mathrm{e}}(\omega) \\ c_{\mathrm{e}}^*(-\omega) \end{pmatrix} = \begin{pmatrix} F_{\mathrm{m}}(\omega) \\ F_{\mathrm{m}}^*(-\omega) \\ F_{\mathrm{e}}(\omega) \\ F_{\mathrm{e}}^*(-\omega) \end{pmatrix} , \tag{9.36}$$

or

$$\begin{pmatrix} c_{\mathrm{m}}(\omega) \\ c_{\mathrm{m}}^*(-\omega) \\ c_{\mathrm{e}}(\omega) \\ c_{\mathrm{e}}^*(-\omega) \end{pmatrix} = \chi(\omega) \begin{pmatrix} F_{\mathrm{m}}(\omega) \\ F_{\mathrm{m}}^*(-\omega) \\ F_{\mathrm{e}}(\omega) \\ F_{\mathrm{e}}^*(-\omega) \end{pmatrix} , \tag{9.37}$$

where the susceptibility $\chi(\omega)$ is given by

$$\chi(\omega) = (J - i\omega)^{-1} . \tag{9.38}$$

For the case where $J = J_0 + \mathcal{G}V$ [see Eq. (9.29)] the susceptibility $\chi(\omega)$ can be expanded as a power series in $\mathcal{G}$ according to

$$\begin{aligned}\chi(\omega) &= (J_0 - i\omega + \mathcal{G}V)^{-1} \\ &= \left((J_0 - i\omega)\left(1 + (J_0 - i\omega)^{-1}\mathcal{G}V\right)\right)^{-1} \\ &= (1 + \mathcal{G}\chi_0(\omega)V)^{-1}\chi_0(\omega) \\ &= \chi_0(\omega) - \mathcal{G}\chi_0(\omega)V\chi_0(\omega) + \mathcal{G}^2\chi_0(\omega)V\chi_0(\omega)V\chi_0(\omega) - \cdots ,\end{aligned} \tag{9.39}$$

where the susceptibility matrix

$$\chi_0(\omega) = (J_0 - i\omega)^{-1} , \tag{9.40}$$

which can be expressed in a block form as

$$\chi_0(\omega) = \left(\begin{array}{c|c} \chi_{\mathrm{m}}(\omega) & 0 \\ \hline 0 & \chi_{\mathrm{a}}(\omega) \end{array}\right) , \tag{9.41}$$

where the mechanical block $\chi_{\mathrm{m}}(\omega)$ is given by

$$\chi_{\mathrm{m}}(\omega) = \begin{pmatrix} (\lambda_{\mathrm{m}} - i\omega)^{-1} & 0 \\ 0 & (\lambda_{\mathrm{m}}^* - i\omega)^{-1} \end{pmatrix} , \tag{9.42}$$

and where the ancilla block $\chi_{\mathrm{a}}(\omega)$ is given by

$$\chi_{\mathrm{a}}(\omega) = (J_{\mathrm{a}} - i\omega)^{-1} , \tag{9.43}$$

represents the response at angular frequency ω of the decoupled system.

9.1.6 Perturbation Theory

The four eigenvalues of $J = J_0 + \mathcal{G}V$ are labeled by λ_1, λ_2, λ_3 and λ_4. In the limit $\mathcal{G} \to 0$, i.e. when the mechanical resonator is decoupled from the ancilla system, it is assumed that $\lambda_1 \to \lambda_{\mathrm{m}}$ and $\lambda_2 \to \lambda_{\mathrm{m}}^*$. When $\mathcal{G}$ is sufficiently small the eigenvalues λ_1 and λ_2, which henceforth are referred to as the mechanical eigenvalues, can be calculated using perturbation theory.

In general, the eigenvalues of the $N \times N$ matrix J ($N = 4$ for the current case of optomechanical cavity) are found by solving

$$(J_0 + \mathcal{G}V)\,|v) = \lambda\,|v) , \tag{9.44}$$

where $|v)$ represents a column eigenvector of $J = J_0 + \mathcal{G}V$ with corresponding eigenvalue λ. The symbol $|n)$ is used to label a unit column vector $|n) = (\sigma_1, \sigma_2, \cdots, \sigma_N)^T$ and the symbol $(n|$ labels a row unit vector $(n| = (\sigma_1, \sigma_2, \cdots, \sigma_N)$, where $\sigma_m = \delta_{nm}$. Let $R(\omega)$ be an $N \times N$ matrix, which in a block form is given by

$$R(\omega) = \left(\begin{array}{c|c} \begin{matrix} 0 & 0 \\ 0 & 0 \end{matrix} & 0 \\ \hline 0 & \chi_{\mathrm{a}}(\omega) \end{array}\right) , \tag{9.45}$$

where the susceptibility matrix $\chi_{\mathrm{a}}(\omega)$ is given by $\chi_{\mathrm{a}}(\omega) = (J_{\mathrm{a}} - i\omega)^{-1}$ [see Eq. (9.43)].

Claim. For the case of high quality factor (i.e. the case where $\gamma_{\mathrm{m}} \ll \omega_{\mathrm{m}}$) the mechanical eigenvalues λ_1 and λ_2 are given by

$$\lambda_1 = i\omega_{\mathrm{m}} + \gamma_{\mathrm{m}} + \mathcal{G}\,(1|\,V\,|1) - \mathcal{G}^2\,(1|\,VR\,(\omega_{\mathrm{m}})\,V\,|1) + O\left(\mathcal{G}^3\right) \ , \tag{9.46}$$

and

$$\lambda_2 = -i\omega_{\mathrm{m}} + \gamma_{\mathrm{m}} + \mathcal{G}\,(2|\,V\,|2) - \mathcal{G}^2\,(2|\,VR\,(-\omega_{\mathrm{m}})\,V\,|2) + O\left(\mathcal{G}^3\right) \ . \tag{9.47}$$

Proof. The eigenvalues λ_n, where $n \in \{1, 2\}$, are found by solving [see Eq. (9.44)]

$$\mathcal{G}V\,|v_n) + (J_0 - \lambda_n)\,|v_n) = 0 \ . \tag{9.48}$$

Multiplying by the $N \times N$ matrix R_n from the left, where R_1 and R_2 in a block form are given by [see Eq. (9.30)]

$$R_1 = \left(\begin{array}{cc|c} 0 & 0 & \\ 0 & (\lambda_{\mathrm{m}}^* - \lambda_1)^{-1} & 0 \\ \hline \multicolumn{2}{c|}{0} & (J_{\mathrm{a}} - \lambda_1)^{-1} \end{array}\right) , \tag{9.49}$$

$$R_2 = \left(\begin{array}{cc|c} (\lambda_{\mathrm{m}} - \lambda_2)^{-1} & 0 & \\ 0 & 0 & 0 \\ \hline \multicolumn{2}{c|}{0} & (J_{\mathrm{a}} - \lambda_2)^{-1} \end{array}\right) , \tag{9.50}$$

yields

$$R_n\mathcal{G}V\,|v_n) + R_n\,(J_0 - \lambda_n)\,|v_n) = 0 \ . \tag{9.51}$$

The following holds

$$R_n\,(J_0 - \lambda_n) = Q_n \ ,$$

where

$$Q_n = 1 - P_n \ , \tag{9.52}$$

and where

$$P_n = |n)\,(n| \tag{9.53}$$

is a projection matrix onto the one dimensional subspace spanned by $|n)$, and thus

$$\mathcal{G}R_nV\,|v_n) + Q_n\,|v_n) = 0 \ . \tag{9.54}$$

With the help of Eqs. (9.52) and (9.54) one finds that

$$|v_n) = (1 + \mathcal{G}R_nV)^{-1} P_n |v_n)$$
$$= \left(1 - \mathcal{G}R_nV + \mathcal{G}^2R_nVR_nV - \mathcal{G}^3R_nVR_nVR_nV + \cdots\right) P_n |v_n) \ . \tag{9.55}$$

In general the eigenvector $|v_n)$ is determined up to multiplication by a constant. For simplicity the constant is chosen such that the following holds

$$P_n |v_n) = |n) \ , \tag{9.56}$$

thus [see Eq. (9.53)]

$$(n |v_n) = 1 \ . \tag{9.57}$$

Multiplying [see Eq. (9.44)]

$$\lambda_n |v_n) = (J_0 + \mathcal{G}V) |v_n) \ , \tag{9.58}$$

from the left by $(n|$ yields

$$(n |v_n) \lambda_n = (n| J_0 |v_n) + \mathcal{G} (n| V |v_n) \ , \tag{9.59}$$

thus with the help of Eqs. (9.55) and (9.57) one finds that

$$\lambda_n = \lambda_{n0} + \mathcal{G} (n| V |n) - \mathcal{G}^2 (n| VR_nV |n) + O\left(\mathcal{G}^3\right) \ , \tag{9.60}$$

where

$$\lambda_{n0} = \begin{cases} \lambda_{\mathrm{m}} & n = 1 \\ \lambda_{\mathrm{m}}^* & n = 2 \end{cases} \ . \tag{9.61}$$

As can be seen from its definition [see Eqs. (9.49) and (9.50)], the matrix R_1 (R_2) depends on the unknown eigenvalue $\lambda_1(\lambda_2)$, which formally can be expanded as a power series in $\mathcal{G}$. Keeping terms up to second order in $\mathcal{G}$ only in Eq. (9.60) allows evaluating the matrix R_1 according to (9.49) and R_2 according to (9.50), where λ_n is substituted by λ_{n0}. Note that the term $(\lambda_{\mathrm{m}}^* - \lambda_1)^{-1}$ in the matrix R_1 [see Eq. (9.49)] do not contribute to the matrix element $(1| VR_1V |1)$, and the term $(\lambda_{\mathrm{m}} - \lambda_2)^{-1}$ in the matrix R_2 [see Eq. (9.50)] do not contribute to the matrix element $(2| VR_2V |2)$. Furthermore, for the case $\gamma_{\mathrm{m}} \ll \omega_{\mathrm{m}}$ the matrix R_1 in Eq. (9.60) can be approximated by $R(\omega_{\mathrm{m}})$ and the matrix R_2 by $R(-\omega_{\mathrm{m}})$.

9.1.7 The Mechanical Eigenvalues

With the help of Eqs. (9.30), (9.31), (9.34), (9.45), (9.46) and (9.47) one finds that the mechanical eigenvalues λ_1 and λ_2 are given by

$$\lambda_1 = i\omega_{\mathrm{m}} + \gamma_{\mathrm{m}} - \mathcal{G}^2\omega_{\mathrm{m}}\omega_{\mathrm{e}} \left(B_{\mathrm{e}}^* \ B_{\mathrm{e}} \right) \chi_{\mathrm{a}} (\omega_{\mathrm{m}}) \begin{pmatrix} -B_{\mathrm{e}} \\ B_{\mathrm{e}}^* \end{pmatrix} + O\left(\mathcal{G}^3\right) \ , \tag{9.62}$$

and

$$\lambda_2 = -i\omega_{\mathrm{m}} + \gamma_{\mathrm{m}} - \mathcal{G}^2\omega_{\mathrm{m}}\omega_{\mathrm{e}} \left(B_{\mathrm{e}}^* \ B_{\mathrm{e}} \right) \chi_{\mathrm{a}} \left(-\omega_{\mathrm{m}} \right) \begin{pmatrix} B_{\mathrm{e}} \\ -B_{\mathrm{e}}^* \end{pmatrix} + O\left(\mathcal{G}^3 \right) \ , \tag{9.63}$$

where

$$\chi_{\mathrm{a}} \left(\pm\omega_{\mathrm{m}} \right) = \begin{pmatrix} \frac{1}{\lambda_{\mathrm{e}} \mp i\omega_{\mathrm{m}}} & 0 \\ 0 & \frac{1}{\lambda_{\mathrm{e}}^* \mp i\omega_{\mathrm{m}}} \end{pmatrix} \ , \tag{9.64}$$

and where $\lambda_{\mathrm{e}} = -i\Delta + \gamma_{\mathrm{e}}$. In terms of the dimensionless detuning d and dimensionless cavity damping rate g, which are given by

$$d = \frac{\Delta}{\omega_{\mathrm{m}}} \ , \tag{9.65}$$

$$g = \frac{\gamma_{\mathrm{e}}}{\omega_{\mathrm{m}}} \ , \tag{9.66}$$

the mechanical eigenvalues can be expressed as

$$\lambda_1 = i\omega_{\mathrm{m}} + \gamma_{\mathrm{m}} + \mathcal{G}^2 E_{\mathrm{e}}\omega_{\mathrm{e}} \Xi \left(d, g \right) + O\left(\mathcal{G}^3 \right) \ , \tag{9.67}$$

and

$$\lambda_2 = \lambda_1^* \ , \tag{9.68}$$

where the function $\Xi \left(d, g \right)$ is given by

$$\begin{aligned} \Xi \left(d, g \right) &= \frac{1}{-i\left(d+1 \right) + g} - \frac{1}{-i\left(-d+1 \right) + g} \\ &= \frac{-4dg - 2id\left(1 - d^2 - g^2 \right)}{\left[\left(1+d \right)^2 + g^2 \right] \left[\left(1-d \right)^2 + g^2 \right]} \ . \end{aligned} \tag{9.69}$$

Note that when cavity nonlinearity is disregarded the variable $E_{\mathrm{e}} = \left| B_{\mathrm{e}} \right|^2$, which is proportional to the energy that is stored in the LC cavity, is given by [see Eq. (9.22)]

$$E_{\mathrm{e}} = \frac{\left| b_{\mathrm{p}} \right|^2}{\omega_{\mathrm{m}}^2 \left(d^2 + g^2 \right)} \ . \tag{9.70}$$

The effective mechanical damping rate $\gamma_{\mathrm{m,eff}} = \operatorname{Re} \lambda_1 = \operatorname{Re} \lambda_2$ to second order in $\mathcal{G}$ is given by [see Eqs. (9.5), (9.6), (9.15), (9.16) and (9.70)]

$$\gamma_{\mathrm{m,eff}} = \gamma_{\mathrm{m}} \left(1 - \beta_{\mathrm{ba}} \frac{1}{d^2 + g^2} \frac{dg}{\left[\left(1+d \right)^2 + g^2 \right] \left[\left(1-d \right)^2 + g^2 \right]} \right) \ . \tag{9.71}$$

where

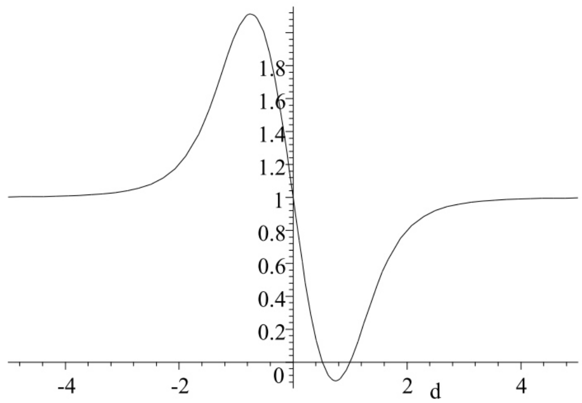


Fig. 9.2. The ratio $\gamma_{\mathrm{m,eff}}/\gamma_{\mathrm{m}}$ calculated according to Eq. (9.71) for the case $\beta_{\mathrm{ba}} = 10$ and $g = 1$.

$$\beta_{\mathrm{ba}} = \frac{C_0 V_1^2}{k d_0^2} \frac{\omega_{\mathrm{m}}}{\gamma_{\mathrm{m}}} \left(\frac{\omega_{\mathrm{e}}}{\omega_{\mathrm{m}}} \right)^3 . \tag{9.72}$$

The ratio $\gamma_{\mathrm{m,eff}}/\gamma_{\mathrm{m}}$ is plotted in Fig. 9.2 for the case $\beta_{\mathrm{ba}} = 10$ and $g = 1$. In this example $\gamma_{\mathrm{m,eff}}$ becomes negative in a range of detuning near $d = 0.75$. In that region the fixed point becomes unstable. The transition between the stable and unstable regions, at which the real part of two eigenvalues of the Jacobian matrix simultaneously vanishes, is called Hopf bifurcation. Typically, near the Hopf bifurcation in the unstable region the system periodically oscillates near the fixed point, which becomes unstable. Such oscillations are commonly called limit-cycle oscillations or self-excited oscillations. The Hopf bifurcation can be either supercritical or subcritical, depending on the sign of the first Lyapunov coefficient, which is proportional to the additive inverse of the effective nonlinear damping of the mechanical resonator. Bistability of the fixed point and the limit cycle occurs in a certain range of parameters when the bifurcation is subcritical, whereas no such bistability occurs when the bifurcation is supercritical.

9.2 Bolometric Optomechanical Coupling

The displacement of a mechanical resonator can be optically detected by illuminating the vibrating resonator with a laser beam and monitoring the off reflected optical power. However, optical absorption by the illuminated mechanical resonator may give rise to heating, which in turn can cause thermal deformation of the suspended mechanical resonator due to mismatch in thermal expansion between the mechanical resonator and its supporting substrate [see Eq. (7.14) and Fig. 8.7]. To model the effect of thermal deformation it is assumed that a temperature dependent force proportional to θT_{R} acts on the mechanical resonator, where $T_{\mathrm{R}} = T - T_0$ is the relative temperature, T is the temperature of the suspended mechanical resonator and T_0 is the base temperature (i.e. the temperature of the supporting substrate). The equation of motion for the mechanical resonator is thus taken to be given by [see Eq. (9.17)]

$$\dot{A}_{\mathrm{m}} + \Theta_{\mathrm{m}} = F_{\mathrm{m}} \ , \tag{9.73}$$

where

$$\Theta_{\mathrm{m}} \left(A_{\mathrm{m}}, A_{\mathrm{m}}^*, T_{\mathrm{R}}\right) = \left(i\omega_{\mathrm{m}} + \gamma_{\mathrm{m}}\right) A_{\mathrm{m}} + i\theta T_{\mathrm{R}} \ , \tag{9.74}$$

The time evolution of the relative temperature T_{R} is governed by the thermal balance equation

$$\dot{T}_{\mathrm{R}} = I - \kappa T_{\mathrm{R}} \ , \tag{9.75}$$

where $I = I(x)$ is proportional to the heating power and κ is the thermal decay rate. For small x, the approximation $I(x) \simeq I_0 + I_1 x$ is employed, where both I_0 and I_1 are assumed to be constants. Thus the thermal balance equation can be rewritten as

$$\dot{T}_{\mathrm{R}} + \Theta_{\mathrm{t}} = F_{\mathrm{t}} \tag{9.76}$$

where [see Eq. (9.11)]

$$\Theta_{\mathrm{t}} \left(A_{\mathrm{m}}, A_{\mathrm{m}}^*, T_{\mathrm{R}}\right) = -I_0 - I_1 \frac{A_{\mathrm{m}} + A_{\mathrm{m}}^*}{\sqrt{2m\omega_{\mathrm{m}}^2}} + \kappa T_{\mathrm{R}} \ . \tag{9.77}$$

The 3×3 Jacobian matrix J is given by

$$J = \partial \left(\Theta_{\mathrm{m}}, \Theta_{\mathrm{m}}^*, \Theta_{\mathrm{t}}\right) / \partial \left(A_{\mathrm{m}}, A_{\mathrm{m}}^*, T_{\mathrm{R}}\right) = J_0 + V \ , \tag{9.78}$$

where

$$J_0 = \begin{pmatrix} \lambda_{\mathrm{m}} & 0 & 0 \\ 0 & \lambda_{\mathrm{m}}^* & 0 \\ 0 & 0 & J_{\mathrm{a}} \end{pmatrix} \ , \tag{9.79}$$

$$V = \begin{pmatrix} 0 & 0 & i\theta \\ 0 & 0 & i\theta \\ -\frac{I_1}{\sqrt{2m\omega_{\mathrm{m}}^2}} & -\frac{I_1}{\sqrt{2m\omega_{\mathrm{m}}^2}} & 0 \end{pmatrix} \ , \tag{9.80}$$

$\lambda_{\mathrm{m}} = i\omega_{\mathrm{m}} + \gamma_{\mathrm{m}}$ and $J_{\mathrm{a}} = \kappa$, and thus the mechanical eigenvalues λ_1 and λ_2 to lowest nonvanishing order in θI_1 are given by [see Eqs. (9.46) and (9.47)]

$$\lambda_1 = i\omega_{\mathrm{m}} + \gamma_{\mathrm{m}} + \frac{I_1\theta}{\sqrt{2m\omega_{\mathrm{m}}^2}} \frac{i\kappa - \omega_{\mathrm{m}}}{\kappa^2 + \omega_{\mathrm{m}}^2} \ , \tag{9.81}$$

and

$$\lambda_2 = \lambda_1^* \ . \tag{9.82}$$

The effective mechanical damping rate $\gamma_{\mathrm{m,eff}} = \operatorname{Re}\lambda_1 = \operatorname{Re}\lambda_2$ to lowest nonvanishing order in θI_1 is thus given by

$$\gamma_{\mathrm{m,eff}} = \gamma_{\mathrm{m}} \left(1 - \frac{I_1 \theta}{\sqrt{2m}\omega_{\mathrm{m}}^2 \gamma_{\mathrm{m}}} \frac{1}{1 + \frac{\kappa^2}{\omega_{\mathrm{m}}^2}} \right) \ . \tag{9.83}$$

9.3 Coupling to Spins

Mechanical resonators can be employed for sensing spin polarization in magnetic materials. In this section back-reaction effects are discussed in a coupled system composed of a mechanical resonator and a driven spin system.

9.3.1 The Decoupled Spin System

The dynamics of the polarization vector $\mathbf{P} = P_x \hat{\mathbf{x}} + P_y \hat{\mathbf{y}} + P_z \hat{\mathbf{z}}$, which describes the state of the spin system, is governed by the Bloch equations

$$\frac{\mathrm{d}\mathbf{P}}{\mathrm{d}t} = \mathbf{P} \times \Omega + \boldsymbol{\gamma} \ , \tag{9.84}$$

where $\Omega(t)$ is the rotation vector, which is proportional to the externally applied magnetic field vector (the factor of proportionality is called the gyromagnetic ratio). The vector

$$\boldsymbol{\gamma} = -\gamma_2 P_x \hat{\mathbf{x}} - \gamma_2 P_y \hat{\mathbf{y}} - \gamma_1 (P_z - P_{z,\mathrm{s}}) \hat{\mathbf{z}} \tag{9.85}$$

represents the contribution of damping, where $\gamma_1 = 1/T_1$ and $\gamma_2 = 1/T_2$ are the longitudinal and transverse relaxation rates, respectively, and where $P_{z,\mathrm{s}}$ is the equilibrium steady state polarization.

Consider the case where the rotation vector $\Omega(t)$ is taken to be given by

$$\Omega(t) = \omega_1 (\cos(\omega t) \hat{\mathbf{x}} + \sin(\omega t) \hat{\mathbf{y}}) + \omega_0 \hat{\mathbf{z}} \ . \tag{9.86}$$

While ω_1 and ω are both assumed to be real constants, ω_0 is allowed to vary in time according to

$$\omega_0 = \omega_{\mathrm{a}} + \omega_{\mathrm{b}} \sin(\omega_{\mathrm{s}} t) \ , \tag{9.87}$$

where ω_{a}, ω_{b} and ω_{s} are all real constants.

Rotating Frame. In terms of the vectors $\hat{\mathbf{u}}_{\pm} = (1/2)(\hat{\mathbf{x}} \pm i\hat{\mathbf{y}})$ the rotation vector $\Omega(t)$ can be expressed as

$$\Omega(t) = \omega_0(t)\,\hat{\mathbf{z}} + \omega_1\left(e^{-i\omega t}\hat{\mathbf{u}}_+ + e^{i\omega t}\hat{\mathbf{u}}_-\right) . \tag{9.88}$$

With the help of the identities $\hat{\mathbf{z}} \times \hat{\mathbf{u}}_{\pm} = \mp i\hat{\mathbf{u}}_{\pm}$, $\hat{\mathbf{u}}_+ \times \hat{\mathbf{u}}_+ = \hat{\mathbf{u}}_- \times \hat{\mathbf{u}}_- = 0$ and $\hat{\mathbf{u}}_+ \times \hat{\mathbf{u}}_- = -i(1/2)\,\hat{\mathbf{z}}$ one finds that [see Eq. (9.84)]

$$\frac{\mathrm{d}P_z}{\mathrm{d}t} = \frac{i\omega_1\left(e^{i\omega t}P_+ - e^{-i\omega t}P_-\right)}{2} - \gamma_1\left(P_z - P_{z,\mathrm{s}}\right) , \tag{9.89}$$

and

$$\frac{\mathrm{d}P_+}{\mathrm{d}t} = -i\omega_0 P_+ + i\omega_1 e^{-i\omega t}P_z - \gamma_2 P_+ . \tag{9.90}$$

Note that $P_- = P_+^*$. By employing the transformation into the rotating frame

$$P_+(t) = e^{-i\int^t \mathrm{d}t'\,\left(\omega_\mathrm{d}+\omega_0(t')\right)} P_{\mathrm{R}+}(t) , \tag{9.91}$$

$$P_-(t) = e^{i\int^t \mathrm{d}t'\,\left(\omega_\mathrm{d}+\omega_0(t')\right)} P_{\mathrm{R}-}(t) , \tag{9.92}$$

where ω_d is a real constant (to be determined later), Eqs. (9.89) and (9.90) become

$$\frac{\mathrm{d}P_z}{\mathrm{d}t} = i\omega_1\frac{\zeta P_{\mathrm{R}+} - \zeta^* P_{\mathrm{R}-}}{2} - \gamma_1\left(P_z - P_{z,\mathrm{s}}\right) , \tag{9.93}$$

and

$$\frac{\mathrm{d}P_{\mathrm{R}+}}{\mathrm{d}t} = i\omega_\mathrm{d}P_{\mathrm{R}+} + i\omega_1\zeta^* P_z - \gamma_2 P_{\mathrm{R}+} , \tag{9.94}$$

where

$$\zeta = \exp\left(i\int^t \mathrm{d}t'\,\left(\omega - \omega_\mathrm{d} - \omega_0(t')\right)\right) . \tag{9.95}$$

The Bloch equations (9.93) and (9.94) can be written in matrix form as

$$\frac{\mathrm{d}}{\mathrm{d}t}\begin{pmatrix} P_{\mathrm{R}+} \\ P_{\mathrm{R}-} \\ P_z \end{pmatrix} + J_\mathrm{a}\begin{pmatrix} P_{\mathrm{R}+} \\ P_{\mathrm{R}-} \\ P_z \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ \gamma_1 P_{z,\mathrm{s}} \end{pmatrix} , \tag{9.96}$$

where

$$J_\mathrm{a} = \begin{pmatrix} -i\omega_\mathrm{d} + \gamma_2 & 0 & -i\omega_1\zeta^* \\ 0 & i\omega_\mathrm{d} + \gamma_2 & i\omega_1\zeta \\ -\frac{i\omega_1\zeta}{2} & \frac{i\omega_1\zeta^*}{2} & \gamma_1 \end{pmatrix} . \tag{9.97}$$

Jacobi-Anger Expansion. With the help of the Jacobi-Anger expansion, which in general can be expressed as

$$\exp\left(iz\cos\theta\right) = \sum_{n=-\infty}^{\infty} i^n J_n\left(z\right) e^{in\theta} , \tag{9.98}$$

where the notation J_n is used to label Bessel functions of the first kind,one finds that

$$\zeta = \sum_{l'=-\infty}^{\infty} i^{l'} J_{l'}\left(\frac{\omega_{\mathrm{b}}}{\omega_{\mathrm{s}}}\right) e^{i\left(\omega-\omega_{\mathrm{a}}+l'\omega_{\mathrm{s}}-\omega_{\mathrm{d}}\right)t} . \tag{9.99}$$

The angular frequency ω_{d} is chosen such that $\omega-\omega_{\mathrm{a}}+l\omega_{\mathrm{s}}-\omega_{\mathrm{d}}=0$ for some integer l, which is chosen such that $|\omega-\omega_{\mathrm{a}}+l\omega_{\mathrm{s}}|$ is minimized. For such a choice the sum contains a stationary term having the smallest possible detuning $|\omega_{\mathrm{d}}|$. Near resonance, i.e. when $|\omega_{\mathrm{d}}|$ is small, the stationary term is expected to have the dominant effect on dynamics of the system, and consequently the factor $\zeta\left(t\right)$ can be approximated by disregarding all other terms $l'\neq l$ and by disregarding the time independent phase factors in Eq. (9.99)

$$\zeta \simeq J_l\left(\frac{\omega_{\mathrm{b}}}{\omega_{\mathrm{s}}}\right) . \tag{9.100}$$

Note that for the case where ω_0 is taken to be a constant $\zeta=1$ and $\omega_{\mathrm{d}}=\omega-\omega_0$.

Modulating $\boldsymbol{\gamma_1 P_{zs}}$. Consider the case where $\gamma_1 P_{z,\mathrm{s}}$ is modulated in time according to

$$\gamma_1 P_{z,\mathrm{s}} = p_0 + e^{-i\omega_{\mathrm{p}}t}p_1 , \tag{9.101}$$

and assume that in steady state the solution can be expressed as

$$\begin{pmatrix} P_{\mathrm{R}+} \\ P_{\mathrm{R}-} \\ P_z \end{pmatrix} = \begin{pmatrix} P_{\mathrm{R}+,0} \\ P_{\mathrm{R}-,0} \\ P_{z,0} \end{pmatrix} + e^{-i\omega_{\mathrm{p}}t} \begin{pmatrix} P_{\mathrm{R}+,1} \\ P_{\mathrm{R}-,1} \\ P_{z,1} \end{pmatrix} , \tag{9.102}$$

where p_0, p_1, $P_{\mathrm{R}+,0}$, $P_{\mathrm{R}-,0}$, $P_{z,0}$, $P_{\mathrm{R}+,1}$, $P_{\mathrm{R}-,1}$, $P_{z,1}$ and ω_{p} are all constants, and where ω_{p} is real. With the help of Eq. (9.96) one finds that

$$\begin{pmatrix} P_{\mathrm{R}+,0} \\ P_{\mathrm{R}-,0} \\ P_{z,0} \end{pmatrix} = \chi_{\mathrm{a}}\left(0\right) \begin{pmatrix} 0 \\ 0 \\ p_0 \end{pmatrix} , \tag{9.103}$$

and

$$\begin{pmatrix} P_{\mathrm{R}+,1} \\ P_{\mathrm{R}-,1} \\ P_{z,1} \end{pmatrix} = \chi_{\mathrm{a}}\left(\omega_{\mathrm{p}}\right) \begin{pmatrix} 0 \\ 0 \\ p_1 \end{pmatrix} \ . \tag{9.104}$$

where [see Eq. (9.43)]

$$\chi_{\mathrm{a}}\left(\omega'\right) = \left(J_{\mathrm{a}} - i\omega'\right)^{-1} \ . \tag{9.105}$$

The susceptibility $\chi_{\mathrm{a}}\left(\omega_{\mathrm{p}}\right)$ is given by (it is assumed that $\zeta^* = \zeta$)

$$\chi_{\mathrm{a}}\left(\omega_{\mathrm{p}}\right) = \frac{\begin{pmatrix} \chi_+\chi_0 - \frac{\chi_1^2}{2} & -\frac{\chi_1^2}{2} & \chi_1\chi_+ \\ -\frac{\chi_1^2}{2} & \chi_-\chi_0 - \frac{\chi_1^2}{2} & -\chi_1\chi_- \\ \frac{\chi_1\chi_+}{2} & -\frac{\chi_1\chi_-}{2} & \chi_-\chi_+ \end{pmatrix}}{\gamma_2\left(\chi_-\chi_+\chi_0 - \frac{\chi_1^2(\chi_-+\chi_+)}{2}\right)} \ , \tag{9.106}$$

where

$$\chi_+ = 1 - \frac{i\left(\omega_{\mathrm{p}} - \omega_{\mathrm{d}}\right)}{\gamma_2} \ , \tag{9.107}$$

$$\chi_- = 1 - \frac{i\left(\omega_{\mathrm{p}} + \omega_{\mathrm{d}}\right)}{\gamma_2} \ , \tag{9.108}$$

$$\chi_0 = \frac{\gamma_1 - i\omega_{\mathrm{p}}}{\gamma_2} \ , \tag{9.109}$$

$$\chi_1 = \frac{i\omega_1\zeta}{\gamma_2} \ , \tag{9.110}$$

and thus

$$\begin{pmatrix} \frac{\gamma_2 P_{\mathrm{R}+,m}}{p_m} \\ \frac{\gamma_2 P_{\mathrm{R}-,m}}{p_m} \\ \frac{\gamma_2 P_{z,m}}{p_m} \end{pmatrix} = \begin{pmatrix} \frac{\chi_1\chi_+}{\chi_-\chi_+\chi_0 - \frac{\chi_1^2(\chi_-+\chi_+)}{2}} \\ -\frac{\chi_1\chi_-}{\chi_-\chi_+\chi_0 - \frac{\chi_1^2(\chi_-+\chi_+)}{2}} \\ \frac{\chi_-\chi_+}{\chi_-\chi_+\chi_0 - \frac{\chi_1^2(\chi_-+\chi_+)}{2}} \end{pmatrix} \ , \tag{9.111}$$

where $m \in \{0, 1\}$ and where $\omega_{\mathrm{p}} = 0$ for $m = 0$. For the case $m = 0$ the above result yields

$$\begin{pmatrix} \frac{P_{\mathrm{R}+,0}}{P_{z,\mathrm{s}}} \\ \frac{P_{\mathrm{R}-,0}}{P_{z,\mathrm{s}}} \\ \frac{P_{z,0}}{P_{z,\mathrm{s}}} \end{pmatrix} = \begin{pmatrix} \frac{\frac{i\omega_1\zeta}{\gamma_2}\left(1+\frac{i\omega_{\mathrm{d}}}{\gamma_2}\right)}{1+\left(\frac{\omega_{\mathrm{d}}}{\gamma_2}\right)^2+\frac{(\omega_1\zeta)^2}{\gamma_1\gamma_2}} \\ \frac{-\frac{i\omega_1\zeta}{\gamma_2}\left(1-\frac{i\omega_{\mathrm{d}}}{\gamma_2}\right)}{1+\left(\frac{\omega_{\mathrm{d}}}{\gamma_2}\right)^2+\frac{(\omega_1\zeta)^2}{\gamma_1\gamma_2}} \\ \frac{1+\left(\frac{\omega_{\mathrm{d}}}{\gamma_2}\right)^2}{1+\left(\frac{\omega_{\mathrm{d}}}{\gamma_2}\right)^2+\frac{(\omega_1\zeta)^2}{\gamma_1\gamma_2}} \end{pmatrix} \ . \tag{9.112}$$

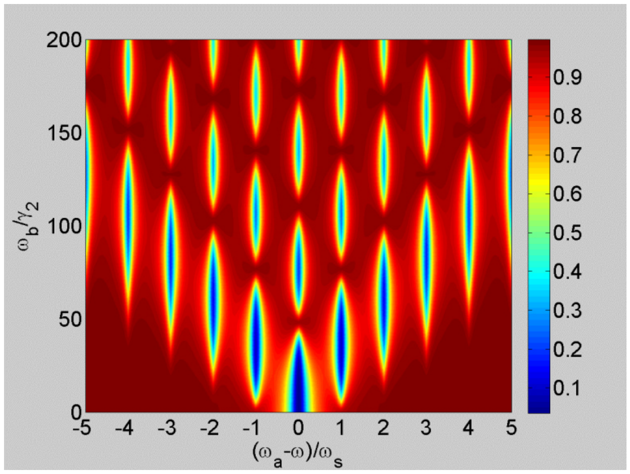


Fig. 9.3. The ratio $P_{z,0}/P_{z,\mathrm{s}}$ as a function of $(\omega_{\mathrm{a}}-\omega)/\omega_{\mathrm{s}}$ and $\omega_{\mathrm{b}}/\gamma_2$ calculated by Eq. (9.114) for the case where $\omega_1/\sqrt{\gamma_1\gamma_2}=5$, $\omega_{\mathrm{s}}/\gamma_2=20$ and $l_0=10$.

As can be see from Eq. (9.112) at resonance, i.e. when $\omega_{\mathrm{d}}=0$, the following holds

$$\frac{(\omega_1\zeta)^2}{\gamma_1\gamma_2}=\frac{P_{z,\mathrm{s}}}{P_{z,0}}-1\ . \tag{9.113}$$

In steady state (for a fixed $\gamma_1 P_{z,\mathrm{s}}$) the ratio $P_{z,0}/P_{z,\mathrm{s}}$ can be calculated using Eq. (9.112). The ratio is numerically calculated using the following expression

$$\frac{P_{z,0}}{P_{z,\mathrm{s}}}=1-\sum_{l=-l_0}^{l_0}\frac{\frac{\omega_1^2 J_l^2\left(\frac{\omega_{\mathrm{b}}}{\omega_{\mathrm{s}}}\right)}{\gamma_1\gamma_2}}{1+\left(\frac{\omega-\omega_{\mathrm{a}}+l\omega_{\mathrm{s}}}{\gamma_2}\right)^2+\frac{\omega_1^2 J_l^2\left(\frac{\omega_{\mathrm{b}}}{\omega_{\mathrm{s}}}\right)}{\gamma_1\gamma_2}}\ . \tag{9.114}$$

Contrary to Eq. (9.100), off resonance terms are not formally neglected. However, their contribution to the sum is expected to be small provided that overlap between neighboring resonances is small. Note also that the sum is truncated with the sum cutoff parameter l_0. The ratio $P_{z,0}/P_{z,\mathrm{s}}$ is plotted in Fig. 9.3 as a function of $(\omega_{\mathrm{a}}-\omega)/\omega_{\mathrm{s}}$ and $\omega_{\mathrm{b}}/\gamma_2$.

9.3.2 The Coupled System

The equations of motion of the coupled system are assumed to be given by

$$\dot{A}_{\mathrm{m}}+\Theta_{\mathrm{m}}=F_{\mathrm{m}}\ , \tag{9.115}$$
$$\dot{P}_{\mathrm{R}+}+\Theta_{\mathrm{R}+}=F_{\mathrm{R}+}\ , \tag{9.116}$$
$$\dot{P}_z+\Theta_z=F_z\ , \tag{9.117}$$

where [see Eqs. (9.17), (9.93) and (9.94)]

$$\Theta_{\mathrm{m}}\left(A_{\mathrm{m}}, A_{\mathrm{m}}^{*}, P_{\mathrm{R}+}, P_{\mathrm{R}-}, P_{z}\right)=\left(i \omega_{\mathrm{m}}+\gamma_{\mathrm{m}}\right) A_{\mathrm{m}}+i g P_{z} , \tag{9.118}$$

$$\begin{aligned} &\Theta_{\mathrm{R}+}\left(A_{\mathrm{m}}, A_{\mathrm{m}}^{*}, P_{\mathrm{R}+}, P_{\mathrm{R}-}, P_{z}\right)=\left(-i \omega_{\mathrm{d}}+\gamma_{2}\right) P_{\mathrm{R}+} \\ &-2 i g\left(A_{\mathrm{m}}+A_{\mathrm{m}}^{*}\right) P_{\mathrm{R}+}-i \omega_{1} \zeta^{*} P_{z} , \end{aligned} \tag{9.119}$$

$$\Theta_{z}\left(A_{\mathrm{m}}, A_{\mathrm{m}}^{*}, P_{\mathrm{R}+}, P_{\mathrm{R}-}, P_{z}\right)=i \omega_{1} \frac{\zeta^{*} P_{\mathrm{R}-}-\zeta P_{\mathrm{R}+}}{2}+\gamma_{1}\left(P_{z}-P_{z, \mathrm{s}}\right) , \tag{9.120}$$

and where g is the coupling constant between the mechanical resonator and the spin system. Due to the coupling a force given by gP_z is applied to the mechanical resonator [see Eq. (9.118)], and the spin angular resonance frequency is shifted by $2g\left(A_{\mathrm{m}}+A_{\mathrm{m}}^{*}\right)$ [see Eq. (9.3.2)].

The 5×5 Jacobian matrix J is given by

$$J=\partial\left(\Theta_{\mathrm{m}}, \Theta_{\mathrm{m}}^{*}, \Theta_{\mathrm{R}+}, \Theta_{\mathrm{R}+}, \Theta_{z}\right) / \partial\left(A_{\mathrm{m}}, A_{\mathrm{m}}^{*}, P_{\mathrm{R}+}, P_{\mathrm{R}-}, P_{z}\right)=J_{0}+g V , \tag{9.121}$$

where the matrix J_0 in a block form is given by [see Eq. (9.97)]

$$J_{0}=\left(\begin{array}{cc|c} \lambda_{\mathrm{m}} & 0 & \multirow{2}{*}{0} \\ 0 & \lambda_{\mathrm{m}}^{*} & \\ \hline \multicolumn{2}{c|}{0} & J_{\mathrm{a}} \end{array}\right) , \tag{9.122}$$

$\lambda_{\mathrm{m}}=i\omega_{\mathrm{m}}+\gamma_{\mathrm{m}}$, and the matrix V is given by

$$V=\left(\begin{array}{ccccc} 0 & 0 & 0 & 0 & i \\ 0 & 0 & 0 & 0 & -i \\ -2 i P_{\mathrm{R}+} & -2 i P_{\mathrm{R}+} & -2 i\left(A_{\mathrm{m}}+A_{\mathrm{m}}^{*}\right) & 0 & 0 \\ 2 i P_{\mathrm{R}-} & 2 i P_{\mathrm{R}-} & 0 & 2 i\left(A_{\mathrm{m}}+A_{\mathrm{m}}^{*}\right) & 0 \\ 0 & 0 & 0 & 0 & 0 \end{array}\right) . \tag{9.123}$$

9.3.3 The Mechanical Eigenvalues

With the help of Eqs. (9.46), (9.47), (9.122) and (9.123) one finds that the mechanical eigenvalues are given by

$$\lambda_{1}=i \omega_{\mathrm{m}}+\gamma_{\mathrm{m}}+\Lambda_{1}+O\left(g^{3}\right) , \tag{9.124}$$

where

$$\Lambda_{1}=2 g^{2}\left(\begin{array}{lll} 0 & 0 & 1 \end{array}\right) \chi_{\mathrm{a}}\left(\omega_{\mathrm{m}}\right)\left(\begin{array}{c} -P_{\mathrm{R}+, 0} \\ P_{\mathrm{R}-, 0} \\ 0 \end{array}\right) , \tag{9.125}$$

and

$$\lambda_2 = \lambda_1^* \,. \tag{9.126}$$

With the help of Eq. (9.106) one obtains

$$\Lambda_1 = -\frac{g^2\chi_1\left(P_{\mathrm{R}+,0}\chi_+ + P_{\mathrm{R}-,0}\chi_-\right)}{\gamma_2\left(\chi_-\chi_+\chi_0 - \frac{\chi_1^2(\chi_-+\chi_+)}{2}\right)} \,. \tag{9.127}$$

Using Eqs. (9.107), (9.108), (9.109), (9.110) and (9.112) one finds that Λ_1 can be written as

$$\Lambda_1 = \frac{2\omega_{\mathrm{m}}\left(\frac{\omega_1\zeta}{\omega_{\mathrm{m}}}\right)^2\frac{\omega_{\mathrm{d}}}{\omega_{\mathrm{m}}}\left(1+\frac{2i\gamma_2}{\omega_{\mathrm{m}}}\right)\frac{g^2P_{z,\mathrm{s}}}{\gamma_2^2+\omega_{\mathrm{d}}^2+\frac{\gamma_2(\omega_1\zeta)^2}{\gamma_1}}}{\frac{\gamma_1}{\omega_{\mathrm{m}}}\frac{\omega_{\mathrm{d}}^2-\omega_{\mathrm{dR}}^2}{\omega_{\mathrm{m}}^2} - i\frac{\omega_{\mathrm{d}}^2-\omega_{\mathrm{dI}}^2}{\omega_{\mathrm{m}}^2}} \,, \tag{9.128}$$

where

$$\frac{\omega_{\mathrm{dR}}}{\omega_{\mathrm{m}}} = \sqrt{1+\frac{2\gamma_2}{\gamma_1}-\left(\frac{\omega_1\zeta}{\omega_{\mathrm{m}}}\right)^2\frac{\gamma_2}{\gamma_1}-\frac{\gamma_2^2}{\omega_{\mathrm{m}}^2}} \,, \tag{9.129}$$

and where

$$\frac{\omega_{\mathrm{dI}}}{\omega_{\mathrm{m}}} = \sqrt{1-\left(\frac{\omega_1\zeta}{\omega_{\mathrm{m}}}\right)^2-\frac{\gamma_2\left(2\gamma_1+\gamma_2\right)}{\omega_{\mathrm{m}}^2}} \,. \tag{9.130}$$

The real [(a) and (c)] and imaginary [(b) and (d)] parts of $\Lambda_1/\omega_{\mathrm{m}}$ are plotted vs. $\omega_{\mathrm{d}}/\omega_{\mathrm{m}}$ and $\omega_1\zeta/\omega_{\mathrm{m}}$ in Fig. 9.4 for the case where $\gamma_1/\omega_{\mathrm{m}} = 0.01$, $\gamma_2/\omega_{\mathrm{m}} = 0.04$ and $g = 0.1$. As can be seen from panels (a) and (b), for this example $|\Lambda_1|$ has two pronounced peaks near the points $\omega_{\mathrm{d}} = \pm\omega_{\mathrm{dI}}$, where ω_{dI} is given by Eq. (9.130). Near these points $\omega_{\mathrm{m}} \simeq \omega_{\mathrm{R}}$, where $\omega_{\mathrm{R}} \equiv \sqrt{(\omega_1\zeta)^2+\omega_{\mathrm{d}}^2}$ is the Rabi frequency of the driven spins. Another peak occurs near $\omega_{\mathrm{d}} = 0$, as can be seen from the plots in panels (c) and (d), in which the region near the point $\omega_{\mathrm{d}} = \omega_1 = 0$ is magnified. However, the peak near $\omega_{\mathrm{d}} = 0$ is much smaller compared to those near $\omega_{\mathrm{d}} = \pm\omega_{\mathrm{dI}}$.

When the peaks at $\omega_{\mathrm{d}} = \pm\omega_{\mathrm{dI}}$ do not overlap and when the small peak near $\omega_{\mathrm{d}} = 0$ can be disregarded, Λ_1 can be expressed to a good approximation in a form similar to Eq. (9.67) [compare with Eq. (9.69)]

$$\Lambda_1 = \left(\frac{g}{\omega_{\mathrm{m}}}\right)^2 E_{\mathrm{s}}\left(-\frac{1}{-i\left(d_{\mathrm{s}}+1\right)+g_{\mathrm{s}}}-\frac{1}{-i\left(-d_{\mathrm{s}}+1\right)+g_{\mathrm{s}}}\right) \,, \tag{9.131}$$

where

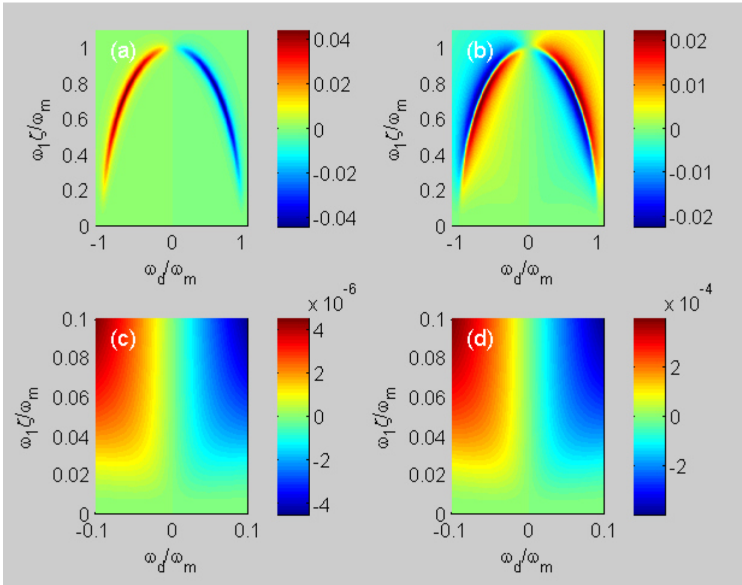


Fig. 9.4. The real [(a) and (c)] and imaginary [(b) and (d)] parts of $\Lambda_1/\omega_{\mathrm{m}}$ [see Eq. (9.128)] for the case where $\gamma_1/\omega_{\mathrm{m}} = 0.01$, $\gamma_2/\omega_{\mathrm{m}} = 0.04$ and $g = 0.1$. The region near the point $\omega_{\mathrm{d}} = \omega_1 = 0$ is magnified in panels (c) and (d) .

$$E_{\mathrm{s}} = \frac{\omega_{\mathrm{m}}\left(\frac{\omega_1\zeta}{\omega_{\mathrm{m}}}\right)^2 \frac{\omega_{\mathrm{d}}\omega_{\mathrm{m}}}{\omega_{\mathrm{dI}}^2} P_{z,\mathrm{s}}}{\left(\frac{\gamma_2}{\omega_{\mathrm{m}}}\right)^2 + \left(\frac{\omega_{\mathrm{d}}}{\omega_{\mathrm{m}}}\right)^2 + \frac{\gamma_2}{\gamma_1}\left(\frac{\omega_1\zeta}{\omega_{\mathrm{m}}}\right)^2} , \tag{9.132}$$

and where

$$d_{\mathrm{s}} = \frac{\omega_{\mathrm{d}}}{\omega_{\mathrm{dI}}} , \tag{9.133}$$

$$g_{\mathrm{s}} = \frac{\gamma_1}{\omega_{\mathrm{m}}} \frac{\omega_{\mathrm{dR}}^2 - \omega_{\mathrm{dI}}^2}{2\omega_{\mathrm{dI}}^2} . \tag{9.134}$$

The expression (9.132) has been derived by assuming that $\gamma_2 \ll \omega_{\mathrm{m}}$.

9.4 Problems

1. Find the eigen frequencies of the system that is described by the Lagrangian (9.4) for the case where $V = 0$.
2. Consider the system that is schematically depicted in Fig. 9.1, and assume that the applied voltage V is a constant. Let x_0 and q_0 be respectively the displacement of the mass and the charge on the capacitor in steady state. Calculate x_0 and q_0 to lowest nonvanishing order in V.

3. In general, the susceptibility matrix $\chi(\omega)$ [see Eq. (9.38)] can be written in a block form as

$$\chi(\omega) = \left(\begin{array}{c|c} \chi_{\mathrm{mm}}(\omega) & \chi_{\mathrm{ma}}(\omega) \\ \hline \chi_{\mathrm{am}}(\omega) & \chi_{\mathrm{aa}}(\omega) \end{array} \right) . \tag{9.135}$$

Calculated the mechanical block $\chi_{\mathrm{mm}}(\omega)$ for the case of optomechanical cavity to lowest nonvanishing order in $\mathcal{G}$.

9.5 Solutions

1. When $V = 0$ the Lagrangian (9.4) to lowest nonvanishing terms in x and q is given by (note that the term $q^2 x/2C_0 d_0$ is disregarded since it is of higher order)

$$\mathcal{L} = \frac{m\left(\dot{x}^2 - \omega_{\mathrm{m}}^2 x^2\right)}{2} + \frac{L\left(\dot{q}^2 - \omega_{\mathrm{e}}^2 q^2\right)}{2} , \tag{9.136}$$

and thus the angular eigen frequencies are ω_{m} and ω_{e}.

2. The values of x and q in steady state, which are labeled by x_0 and q_0 respectively, are found by seeking a stationary solution of the equations of motion (9.7) and (9.8)

$$\omega_{\mathrm{m}}^2 x_0 = \frac{q_0^2}{2md_0C_0} , \tag{9.137}$$

and

$$\omega_{\mathrm{e}}^2 q_0 \left(1 - \frac{x_0}{d_0}\right) = -\frac{V}{L} . \tag{9.138}$$

To lowest nonvanishing order in V the solution is given by

$$x_0 = \frac{C_0 V^2}{2m\omega_{\mathrm{m}}^2 d_0} = \frac{\varepsilon_0 A V^2}{2kd_0^2} , \tag{9.139}$$

and

$$q_0 = -\frac{V}{L\omega_{\mathrm{e}}^2} = -C_0 V = -\frac{\varepsilon_0 A V}{d_0} . \tag{9.140}$$

3. With the help of Eqs. (9.30), (9.34) and (9.39) one finds that

$$\begin{aligned}\chi_{\mathrm{mm}}(\omega) &= \begin{pmatrix} (\lambda_{\mathrm{m}} - i\omega)^{-1} & 0 \\ 0 & (\lambda_{\mathrm{m}}^* - i\omega)^{-1} \end{pmatrix} \\ &\quad + \mathcal{G}^2 \omega_{\mathrm{m}} \omega_{\mathrm{e}} E_{\mathrm{e}} \left(\frac{1}{\lambda_{\mathrm{e}}^* - i\omega} - \frac{1}{\lambda_{\mathrm{e}} - i\omega} \right) \begin{pmatrix} \frac{1}{(\lambda_{\mathrm{m}} - i\omega)^2} & \frac{1}{(\lambda_{\mathrm{m}} - i\omega)(\lambda_{\mathrm{m}}^* - i\omega)} \\ -\frac{1}{(\lambda_{\mathrm{m}} - i\omega)(\lambda_{\mathrm{m}}^* - i\omega)} & \frac{1}{(\lambda_{\mathrm{m}}^* - i\omega)^2} \end{pmatrix} \\ &\quad + O\left(\mathcal{G}^3\right) .\end{aligned} \tag{9.141}$$

References

1. Ref

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