Musical tonality and synchronization

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The current study is motivated by some observations of highly nonlinear dynamical effects in biological auditory systems [1]. We examine the hypothesis that one of the underlying mechanisms responsible for the observed nonlinearity is self-excited oscillation (SEO). According to this hypothesis the detection and processing of input audio signals by biological auditory systems is performed by coupling the input signal with an internal element undergoing SEO. Under appropriate conditions such coupling may result in synchronization between the input signal and the SEO. In this paper we present some supporting evidence for this hypothesis by showing that some well-known phenomena in musical tonality can be explained by the Hopf model of SEO [2] and the Arnold model of synchronization [3]. Moreover, some mathematical properties of these models are employed as guidelines for the construction of some modulations that can be applied to a given musical composition. The construction of some intriguing patterns of musical harmony is demonstrated by applying these modulations to known musical pieces [4].

I. INTRODUCTION

Great deal of evidence supports the hypothesis that the dynamics of some essential elements in the auditory system is highly nonlinear [1]. One of the most convincing evidence comes from measurements of spontaneous otoacoustic emissions (OAE) that is produced by the ear [5, 6]. Nonlinear frequency mixing has been observed in measurements of distortion products of OAE that are evoked using a pair of primary tones [7, 8]. Highly nonlinear response of a living chinchilla's cochlea has been detected using laser velocimetric measurements [9]. Moreover, nonlinearity can be demonstrated in experiments studying the perception of musical harmony, in which human listeners are given an assignment based on a given sound that is played to them [10–12]. In particular, it was shown that the measured perceived pitch of a sound having missing fundamental tones reveals a nonlinear process of frequency mixing in the auditory pathway [13, 14].

While the above-discussed observations clearly demonstrate nonlinear dynamical effects in the auditory system, further study is needed in order to quantitatively explore the underlying mechanisms responsible for these effects. It was suggested that the Hopf bifurcation model for self-excited oscillation (SEO) can be used for the description of the auditory process of hearing [15]. In particular, it was shown that the experimentally observed dynamic range compression [16] can be related to the Hopf model. Moreover, it was suggested that the same Hopf model can be used to describe the underlying physiological mechanisms responsible for some universal (i.e. culture-independent) phenomena in musical tonality [17–19]. In this description, SEO are internally generated in parts of the human brain responsible for audio processing of music.

In this work we further explore possible connections between the Hopf model and musical tonality. The Hopf model provides a generic description of SEO generation, and it allows studying the process of synchronization of SEO to externally applied signals [20–22]. We consider a possible interpretation of some known phenomena in musical harmony, and show that a connection between the Hopf model and musical tonality is revealed by such an interpretation. To further validate this interpretation, some harmonic modulations are defined based on symmetrical properties of the process of synchronization. The creation of intriguing harmonies by applying these modulations to some well known musical compositions is demonstrated [4].

II. SYNCHRONIZATION OF SEO

Some properties of the Hopf model that are potentially relevant to musical tonality are briefly reviewed below. Consider a one-dimensional oscillator evolving in time according to [23]

$$\dot{A} + (\Gamma_{\mathrm{eff}} + i\Omega_{\mathrm{eff}})\, A = \xi(t) + \vartheta(t) \ , \qquad (1)$$

where the complex amplitude A is related to the coordinate $x(t)$ of the oscillator by $x = \operatorname{Re} A$ and overdot denotes a derivative with respect to time t. To lowest nonvanishing order in $|A|^2$ the damping rate Γ_{eff} and the angular resonance frequency Ω_{eff} (both Γ_{eff} and Ω_{eff} are real) are given by $\Gamma_{\mathrm{eff}} = \Gamma_0 + \Gamma_2 |A|^2$ and $\Omega_{\mathrm{eff}} = \Omega_0 + \Omega_2 |A|^2$. The term $\xi(t)$ represents an externally applied force, and the fluctuating term $\vartheta(t)$ represents white noise [24, 25]. In the absence of externally applied force, i.e. when $\xi(t) = 0$, the equation of motion (1) describes a van der Pol oscillator [26]. Consider the case where $\Gamma_2 > 0$, for which a supercritical Hopf bifurcation occurs when the linear damping coefficient Γ_0 vanishes. Above threshold, i.e. when Γ_0 becomes negative, the amplitude $|A|$ of SEO is given by $r_0 = \sqrt{-\Gamma_0/\Gamma_2}$ and the angular frequency Ω_{H} of SEO by $\Omega_{\mathrm{H}} = \Omega_{\mathrm{eff}}(r_0)$. Note that to a good approximation the dependency of Ω_{H} on $|A|$ can be disregarded provided that $|\Omega_2| \ll \Omega_0/r_0^2$. In what follows it will be assumed that this dependency can be disregarded.

While the phase of SEO [27–30] randomly diffuses in time when $\xi(t) = 0$, phase locking [26, 31–35] may occur when forcing is periodically applied. Such locking results in entrainment [36], i.e. synchronization [20–22] between the SEO and the external forcing term $\xi(t)$ [37].

Consider the case of a monochromatic forcing at angular frequency $\omega_d = (1+\alpha)\Omega_H$ and amplitude ω_a, for which ξ is given by $\xi = \omega_a r_0 e^{-i\omega_d t}$. The variable ϕ, which is defined by $\phi = A_\theta + \Omega_H t$, where $A = |A| e^{iA_\theta}$, represents the phase of the oscillator in a frame rotating at angular frequency Ω_H. Let $2\pi(Q_n - n\alpha)$ be the value of the relative phase ϕ at time $t_n = 2\pi n/\Omega_H$, i.e. after n periods of SEO, where n is integer. For the case where the change in ϕ over a single mechanical period $2\pi/\Omega_H$ is small, the evolution of Q_n can be described by the so-called circle map, which for the current case is given by

$$Q_{n+1} = Q_n + \alpha - \frac{K \sin(2\pi Q_n)}{2\pi} , \tag{2}$$

where $K = 2\pi\omega_a/\Omega_H$.

The winding number W is defined by $W = \lim_{n\to\infty}(Q_{n+1} - Q_1)/n$ [34, 38, 39]. For the case of a limit cycle (i.e. phase locking), the winding number is a rational number given by $W = p/q$, where q is the period of the cycle and p is the number of sweeps through the unit interval $[0,1]$ in a cycle when the mapping (2) is considered as modulo 1. Regions of phase locking in the plane that is spanned by the forcing parameters (normalized amplitude K and detuning α) are commonly called Arnold tongues [3] (see Fig. 1). Note that the graph of the function $W(\alpha)$ forms a structure known as a devil's staircase (which is incomplete for $K < 1$, and becomes complete when $K = 1$). In the limit $K \to 0$ the width Δ_α of the region of phase locking corresponding to a given rational value of $W = p/q$, where p and q are relatively prime integers, is given by [3]

$$2\pi\Delta_\alpha \simeq K^q . \tag{3}$$

As can be seen from Eq. (2, the winding number W satisfies the following symmetry relation

$$W(\alpha) = W(1-\alpha) . \tag{4}$$

The spectral density of the amplitude of oscillation $x(t) = \operatorname{Re} A(t)$ has some universal properties near the transition between the locked and the unlocked regions [41]. These properties are briefly described below for the case of the primary Arnold tongue, i.e. for the region $\omega_d \simeq \Omega_H$, for which the spectral density near the transition can be analytically evaluated. Phase locking in this region occurs when $|i_b| \leq 1$, where $i_b = 2\pi\alpha/K = (\omega_d - \Omega_H)/\omega_a$. Outside the locking region where $|i_b| > 1$, on the other hand, the spectrum contains sidebands at the angular frequencies $\omega_d + m\omega_s$, where m is an integer, the sideband spacing ω_s is given by $\omega_s = \omega_a\sqrt{i_b^2 - 1}$, and the average frequency ω_{ave} in this region is given by $\omega_{ave} = \omega_d - \omega_a\sqrt{i_b^2 - 1}$. It was shown in Ref. [42] that similar sidebands (with similar dependency of the average frequency on the detuning) occur near the edge of other Arnold tongues.

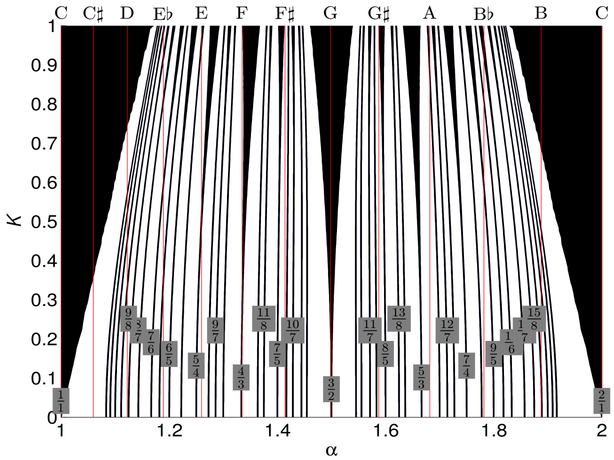

FIG. 1: Arnold tongues in the plane of normalized amplitude K and detuning α (only tongues corresponding to rational values p/q with $q < 13$ are drawn). The red lines indicate relative frequencies of the 12 notes of the μ_2 musical system.

k	d	f'	δ_k	**note**	**interval**
0	0	1	0	C	perfect unison (tonic)
7	1	3/2	0.041	G	perfect fifth (dominant)
5	−1	4/3	0.11	F	perfect fourth (vice dominant)
9	3	5/3	0.25	A	major sixth
4	4	5/4	0.32	E	major third
3	−3	6/5	0.40	E♭	minor third
8	−4	8/5	0.42	G♯	minor sixth
10	−2	7/4	0.42	B♭	minors seventh
6	6	7/5	0.43	F♯	diminished fifth, augmented fourth
2	2	9/8	0.47	D	major second
11	5	15/8	0.58	B	major seventh
1	−5	16/15	0.72	C♯	minor second

TABLE I: Sorting the musical intervals of the μ_2 system according to their effective harmonic detunings δ_k. The obtained ordering is almost identical to the one obtained from the measure of musical consonances (or perfection) that was suggested by Helmholtz in 1885 [40].

When $|i_b| \gtrsim 1$, i.e. just outside the locking region, the sidebands give rise to relatively strong beats. In musical tonality, beats commonly generate a dissonant [40, 43, 44]. On the other hand the region of synchronization is assumed to be associated with a consonant. These musical tonality effects are demonstrated by a media file, which can be downloaded from [4]. The audio in this file is generated from the solution of Eq. (1) in the region $\omega_d \simeq \Omega_H$ [42]. The plot shows the spectrum for a variety of values of the normalized frequency detuning i_b, both below and above the threshold of synchronization occurring at $|i_b| = 1$. The dissonant nature of the sound in the region $|i_b| \gtrsim 1$ is apparent.

III. MUSICAL SYSTEMS

With a relatively weak forcing, synchronization is possible only when the frequency ratio α (between forcing frequency and SEO frequency) is sufficiently close to a rational value. A possible connection between this property and some phenomena in musical tonality is discussed below.

A musical system with N notes per octave is henceforth referred to as N notes system. The frequency of the k'th note, where $k \in \{0, 1, 2 \cdots, N-1\}$, is denoted by $F_{\mathrm{T}} f_k$, where F_{T} is the frequency of a reference note called the tonic tone. For an equal temperament musical tuning the relative frequencies f_k are equally spaced on a logaritmic scale, i.e. f_k is given by $f_k = 2^{k/N}$. For this tuning method the frequency ratio between the notes $k+1$ and k is a constant independent on k.

For a given integer $h \in \{1, 2, 3, \cdots\}$ a musical system denoted as μ_h can be constructed according to the following procedure. The rational number η_h is defined to be a ratio of two adjacent primes $\eta_h = p_h/p_{h-1}$, where p_h denotes the h'th element of the set of prime numbers $\{2, 3, 5, 7, 11, 13, \cdots\}$ and where $p_0 = 1$. The integer N_h is the number of notes per octave of the musical system μ_h. The notes in a given octave are labeled by a key number $k = 0, 1, 2 \cdots, N_h - 1$. The number N_h is chosen such that the set of relative frequencies $\left\{f_k = 2^{k/N_h}\right\}$, where $k \in \{0, 1, 2 \cdots, N_h - 1\}$, contains an element, whose key number k is denoted by M_h, for which $f_{M_h} = 2^{M_h/N_h} \simeq \eta_h$. In general, exact match between the relative frequency f_{M_h} of the $k = M_h$ note (which can be irrational) and the rational number η_h cannot be obtained with integer finite values for both N_h and M_h. In practice, a mismatch of about 0.1% or less is musically acceptable. Note that the condition $f_{M_h} \simeq \eta_h$ implies that $f_{N_h - M_h} \simeq 2/\eta_h$, i.e. the note with key number $k = N_h - M_h$ has a relative frequency close to $2/\eta_h$ with respect to the tonic. The note with key number M_h ($N_h - M_h$) is henceforth referred to as the dominant (vice dominant), and the following holds $M_h \simeq \log_2 (\eta_h)^{N_h}$. Note that the matching condition $f_{M_h} \simeq \eta_h = p_h/p_{h-1}$ can be rephrased as the requirement that the p_h'th overtone of the tonic is close to the p_{h-1}'th overtone of the dominant (in general, the frequency of the m'th overtone of a note having frequency f is mf).

For the simplest musical system μ_1 the ratio η_1 is given by $\eta_1 = 2$. The matching condition between $f_{M_1} = 2^{M_1/N_1}$ and η_1 is exactly satisfies for this case with the integers $N_1 = 1$ and $M_1 = 1$. With these integers the musical system μ_1 having a single note per octave is constructed.

The second system μ_2, which is based on the rational number $\eta_2 = 3/2$, is the common western 12 notes per octave musical system. The integer number N_2 is obtained from the requirement that $\log_2 (3/2)^{N_2}$ is close to an integer. The choice of the integer $N_2 = 12$ and the corresponding dominant key number $M_2 = 7$ and vice dominant key number $N_{12} - M_2 = 5$ is based on the relation $\log_2 (3/2)^{12} \simeq 7.0196$. For these integers $\eta_2/f_{M_2} \simeq 1.0011$, thus the matching condition is satisfied to within an error of about 0.1%.

The third musical system μ_3 is based on the rational number $\eta_3 = 5/3$. The number of notes per octave $N_3 = 19$ and the corresponding dominant key number $M_3 = 14$ and vice dominant key number $N_3 - M_3 = 5$ are chosen based on the relation $\log_2 (5/3)^{19} \simeq 14.002$, and the following holds $\eta_3/f_{M_3} \simeq 5/3$ to within an error of about 0.01%.

Note that an alternative procedure based on the golden mean for the construction of musical scales having N notes per octave has been proposed in [45]. For each value of N the musical quality of the generated scale is quantified by the so-called mean quadratic dispersion [45]. Even though the construction procedure proposed in [45] seems unrelated to the one described above, quite remarkably, both procedures reveal that the lowest non-trivial (i.e. larger than unity) 'good' values of N are 12 and 19.

Consider a musical system μ characterized by the integers N and M. When N and M are relatively prime (i.e. the only positive integer that divides both is 1) it is convenient to specify to any given note having a key number k another integer called the index number d, which is defined by the congruence relation $k \equiv Md \pmod{N}$. The inverse congruence relation reads $d \equiv M^{-1}k$ (mod N), where the integer M^{-1} is the so-called modular multiplicative inverse. Note that $M_2^{-1} = 7$ for the musical system μ_2 and $M_3^{-1} = 15$ for the musical system μ_3. Both the key number k and the index number d specify the interval between a given note and the tonic. However, the index number d measures this interval using the interval between the dominant and the tonic as a unit step (e.g. the index number of the dominant note is $d = 1$ and the index number of the vice dominant note is $d = -1$). Note that the approximation $f_{k=M} \simeq \eta$ can be employed in order to express all N relative frequencies f_k in terms of η and d as

$$\log_2 f_k \simeq \left(\log_2 \eta^d\right) \pmod{1} . \tag{5}$$

IV. HARMONIC DETUNING

While equal temperament musical tuning commonly gives rise to irrational frequency ratios, synchronization (with a relatively weak signal) is most efficient with rational values of ratios of small integers (i.e. rational numbers of relatively low hierarchy in the so-called Farey tree). The level of irrationality can be measured in a variety of ways, including the so-called Liouville Roth irrationality exponent [46, 47]. Motivated by Eq. (3) for the asymptotic width of the Arnold tongues, an alternative measure of irrationality is employed. The inaccuracy corresponding to an approximation of a given frequency f by a rational value p/q, where p and q are relatively prime integers, is quantified using the so-called effective harmonic detuning $D_{p/q}(f)$, which is defined by $D_{p/q}(f) = |f - p/q|^{1/q}$. The infimum harmonic detuning $D(f)$ is defined by $D(f) = \inf_{p/q \in Q} D_{p/q}(f)$, where Q is the set of rational numbers. While $D(f) = 0$ for any $f \in Q$, the infimum harmonic detuning $D(f)$ can take finite positive values for irrational $f \notin Q$.

	S	**A**	**I**	**AI**	**T**
Bach C	V	V	V	V	V
Mozart LD	V	V	V	V	

TABLE II: Musical modulations are demonstrated using the prelude in C major by Bach with the Ave Maria melody added by Charles Gounod (‘Bach C’), and using Mozart Laudate Dominum (‘Mozart LD’). In the columns’ titles ‘S’ stands for source, ‘A’ for the dominant to vice-dominant modulation, ‘I’ for the inter-system modulation from μ_2 (12 notes per octave) to μ_3 (19 notes per octave), ’AI’ for concatenation of ’A’ and ’I’ modulations and ‘T’ for the $f \to 1 - f$ modulation. Red colored piano keys represent notes that are transformed to notes detached from the μ_2 system. A variety of SoundFont files have been used for synthesizing the sound tracks. All media files can be downloaded from [4].

In the so-called just intonation tuning the irrational relative frequencies $f_k = 2^{k/N}$ of the equal temperament tuning are replaced by rational numbers denoted by f'_k. The corresponding effective harmonic detunings are denoted by $\delta_k = D_{f'_k}(f_k)$. Table I presents the calculated values of δ_k for the case of the μ_2 system sorted from small to large. The first and second columns display the values of k and d, respectively, the rational numbers f'_k are indicated in the third column, note names are indicated in the fifth column (C is chosen to be the tonic note), and the names of the corresponding musical intervals are given in the sixth column. This sorting suggests that the value δ_k can provide a useful measure for the relative consonance level of a given musical interval (note that the term *perfect* is used only for the intervals in the top 3 rows and that major and minor triads can be constructed using the intervals in the top 6 rows). Note that a very similar ordering of musical intervals has been obtained from an alternative sorting method based on a model of coupled neural oscillators (see table 1 in [48]).

As can be seen from Fig. 1, with finite amplitude K the center of the p/q Arnold tongue may shift from the point $\alpha = p/q$. Consequently, the effective harmonic detuning cannot provide a reliable measure of the harmonic importance of a given musical interval unless K is sufficiently small. Note that in some cases other considerations may affect the level of consonance. One example is the process of frequency mixing, which may generate a tone at a frequency f_{m} when two input tones at frequencies f_1 and f_2 are played together, where $f_{\mathrm{m}} = l_1 f_1 + l_2 f_2$, and both l_1 and l_2 are integers. For example, for the case of the system μ_2 having $N = 12$ tones per octave, the relation $12 \log_2 \left(1 \times 2^{0/12} + 1 \times 2^{2/12}\right) = 13.029$ suggests that a note similar to C♯ can be generated due to nonlinearity when the notes C and D are played together.

A similar analysis for the case of the μ_3 system with $N = 19$ notes per octave reveals that the lowest 6 values of the effective harmonic detunings are given by $\delta_0 = D_1(1) = 0$ (tonic), $\delta_{14} = D_{5/3}(f_{14}) = 0.052$ (dominant), $\delta_{11} = D_{3/2}(f_{11}) = 0.079$, $\delta_5 = D_{6/5}(f_5) = 0.16$ (vice dominant) and $\delta_8 = D_{4/3}(f_8) = 0.18$. Note that the relatively low values of δ_{11} ($f_{11} \simeq 3/2$, $d = 13$) and δ_8 ($f_8 \simeq 4/3$, $d = 19 - 13 = 6$) can be exploited for playing intervals in the μ_3 system, which are very similar to the dominant and vice dominant intervals of the μ_2 system. Moreover, a general interval in the μ_2 system having key number k and index number $d \equiv M_2^{-1} k \pmod{12}$, where $M_2^{-1} = 7$, can be imitated by playing a note having key number k' in the μ_3 system, where $k' \equiv 11 \times d \pmod{19}$.

V. MUSICAL MODULATIONS

The description of the synchronization process by the Hopf and Arnold models reveals some underlying symmetries. These symmetries can be used as guidelines for the construction of some modulations that can be applied to a given musical piece. Some examples are discussed below. The first one (dominant to vice-dominant modulation) is an intrasystem modulation, which is demonstrated for the μ_2 musical system. The second one is an inter-system modulation, which is demonstrated by converting music from the μ_2 system with 12 notes per octave into the μ_3 system with 19 notes per octave. In the third example (the modulation $f \to 1 - f$) notes in a given system may be converted into notes detached from any equal temperament system. The musical modulations are demonstrated using the prelude in C major by Bach and the Laudate Dominum by Mozart (see table II). Media files presenting these demonstrations can be downloaded from [4].

A. Dominant to vice-dominant modulation

As was pointed out above, the relative frequencies of both the dominant and vice-dominant notes have a relatively small effective harmonic detuning. The so-called dominant to vice-dominant modulation is performed by replacing the dominant frequency η by the vice-dominant frequency $2/\eta$ in Eq. (5). The corresponding transformation of the index number d is given by $d \to d' = A_0(d)$, where $A_\Delta(d) = 2\Delta - d$. Note that the transformation A_Δ mirror reflects d around the point Δ. Moreover, the following holds $A_\Delta^{-1}(d) = A_\Delta(d)$ and $A_{\Delta_2}(d) - A_{\Delta_1}(d) = 2(\Delta_2 - \Delta_1)$. Note that in the μ_2 system with 12 notes per octave the modulation $A_{1/2}(d)$ maps the Lydian mode to the Phrygian mode, the Ionian (major) mode to the Aeolian (minor) mode and the Mixolydian mode to the Dorian mode (and vice versa) without changing the tonic.

B. Intersystem modulation

Replacing the dominant by another frequency having a relatively small effective harmonic detuning can be used also for the construction of intersystem modulations. Consider the modulation $d \rightarrow d'$ from a source musical system having N notes per octave to a target system having $N' > N$ notes per octave, where $d' = d$ for $0 \leq d \leq N/2$ and $d' = N' - N + d$ for $N/2 < d \leq N - 1$. Note that this modulation employs only N out of the $N' > N$ notes per octave in the target system. On a logaritmic scale, this modulation represents a frequency multiplication by the factor $\log \eta' / \log \eta$, where η (η') is the relative frequency of the dominant of the source (target) system [see Eq. (5)]. A modulation from the μ_2 source system having 12 notes per octave to the μ_3 target system having 19 notes per octave is demonstrated by the media files in table II.

C. The modulation $f \rightarrow 1 - f$

The frequency modulation $f \rightarrow f' = T_{f_S}(f) = f_S - f$, where f_S is the SEO frequency, is based on the symmetry relation (4). Note that this modulation may generate notes detached from the musical system. Harmonically satisfying results cannot be commonly obtained with a fixed value of f_S that is kept unchanged throughout the entire musical piece. The varying value of f_S is indicated in the media files demonstrating this modulation (see table II).

VI. CONCLUSIONS

Dictionary definitions of the terms harmony and consonant commonly use words such as pleasing and agreeable, whereas the words harsh and unresolved are used to define the term dissonant. The possible connection between music tonality and SEO suggests alternative definitions that use the term synchronizability. The synchronizability attribute can be used to quantify the complexity of a musical piece. Catchy music has a high level of synchronizability. On the other hand, the learning process that makes a given composition becoming synchronizable is challenging for an unfamiliar music having high level of complexity. Further study is needed in order to explore other implications of synchronization on audio processing in the brain. This may help revealing the encoding and decoding mechanisms that are employed for audio memory and audio recognition.

VII. ACKNOWLEDGMENTS

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