Stability of the Grabert master equation

Eyal Buks and Dvir Schwartz

Andrew and Erna Viterbi Department of Electrical Engineering, Technion, Haifa 32000, Israel

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We study the dynamics of a quantum system having Hilbert space of finite dimension $d_{\rm H}$. Instabilities are possible provided that the master equation governing the system's dynamics contain nonlinear terms. Here we consider the nonlinear master equation derived by Grabert. The dynamics near a fixed point is analyzed by using the method of linearization, and by evaluating the eigenvalues of the Jacobian matrix. We find that all these eigenvalues are non-negative, and conclude that the fixed point is stable. This finding raises the question: under what conditions instability is possible in a quantum system having finite $d_{\rm H}$?

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Consider a given closed quantum system having Hilbert space of finite dimension $d_{\rm H}$, whose master equation, which governs the time evolution of the reduced density matrix ρ , can be expressed as $d\rho/dt = \Theta(\rho) =$ $\Theta_{\rm u}(\rho) - \Theta_{\rm d}(\rho)$. The first term, which is given by $\Theta_{\rm u}(\rho) = (i/\hbar)[\rho, \mathcal{H}],$ where $\mathcal{H} = \mathcal{H}^{\dagger}$ is the Hamiltonian of the closed system, represents unitary evolution, and the second one $\Theta_{d}(\rho)$ represents the effect of coupling between the closed system and its environment. While it is commonly assumed that both the unitary term $\Theta_{\rm u}(\rho)$ and the damping term $\Theta_{d}(\rho)$ are linear in ρ [1, 2], in some cases the master equation can become nonlinear. Two types of nonlinearity are considered below [3, 4]. For the first one, which is henceforth referred to as unitary nonlinearity, the unitary term $\Theta_{\rm u}(\rho)$ is replaced by a nonlinear term. In most cases, unitary nonlinearity originates from either the mean field approximation [5-8], or from a transformation mapping the Hilbert space of finite dimension $d_{\rm H}$ into a space having infinite dimensionality (e.g. the Holstein-Primakoff transformation [9], which can yield a parametric instability in ferromagnetic resonators [10]). Here, we consider the second type, which is henceforth referred to as damping nonlinearity, and focus on the master equation that was proposed by Grabert [11], which has a damping term $\Theta_{d}(\rho)$ nonlinear in ρ .

Grabert has shown that the invalidity of the quantum regression hypothesis gives rise to damping nonlinearity [11]. The nonlinear term added to the master equation ensures that the purity $\text{Tr }\rho^2$ does not exceed unity [12, 13], and that entropy is generated at a non-negative rate, as is expected from the second law of thermodynamics [14]. Note, however, that under appropriate conditions, nonlinear dynamics may allow for faster than light signaling [15].

The Grabert master equation (GME) has a fixed point given by

$$\rho_0 = \frac{e^{-\beta \mathcal{H}}}{\operatorname{Tr}\left(e^{-\beta \mathcal{H}}\right)} \,, \tag{1}$$

where $\beta = 1/(k_{\rm B}T)$ is the inverse of the thermal energy [11]. At the fixed point ρ_0 the system is in thermal equilibrium having Boltzmann distribution.

Here we explore the stability of this fixed point ρ_0 for

the case where the Hamiltonian \mathcal{H} of the closed system is time-independent. In a basis of energy eigenstates of a time-independent Hamiltonian both matrices $\mathcal{H} =$ diag $(E_1, E_2, \dots, E_{d_{\mathrm{H}}})$ and $\rho_0 = \text{diag}(\rho_1, \rho_2, \dots, \rho_{d_{\mathrm{H}}})$ are diagonal, where $\rho_n = e^{-\beta E_n} / \operatorname{Tr}(e^{-\beta \mathcal{H}})$ [see Eq. (1)].

For the case of thermal equilibrium, one may argue that the stability of ρ_0 is obvious. However, the stability of a driven system is anything but obvious. Note that in many cases the rotating wave approximation (RWA) is employed in order to model the dynamics of a given system under external driving, using a transformation into a rotating frame, in which the Hamiltonian becomes timeindependent in the RWA. Thus, our conclusion, that the fixed point ρ_0 is stable for any time independent Hermitian \mathcal{H} , can be extended beyond the limits of thermal equilibrium.

The GME for the reduced density matrix ρ can be expressed as [12]

$$\frac{\mathrm{d}\rho}{\mathrm{d}t} = \Theta\left(\rho\right) = \Theta_{\mathrm{u}}\left(\rho\right) - \Theta_{\mathrm{d}}\left(\rho\right) \,, \tag{2}$$

where the damping term is given by $\Theta_{\rm d}(\rho) = \Theta_{\rm A}(\rho) + \Theta_{\rm B}(\rho)$, where $\Theta_{\rm A}(\rho)$, which is given by $\Theta_{\rm A}(\rho) = \gamma_{\rm E}[Q, [Q, \rho]]$, is linear in ρ , and $\Theta_{\rm B}(\rho)$, which is given by $\Theta_{\rm B}(\rho) = \beta \gamma_{\rm E} \left[Q, [Q, \mathcal{H}]_{\rho}\right]$ is nonlinear. The constant $\gamma_{\rm E} > 0$ is a damping rate, the Hermitian operator $Q^{\dagger} = Q$ describes the interaction between the quantum system and its environment, and

$$A_{\rho} = \int_0^1 \mathrm{d}\eta \; \rho^{\eta} A \rho^{1-\eta} \;. \tag{3}$$

Alternatively, the damping term $\Theta_{\rm d}(\rho)$ can be expressed as $\Theta_{\rm d}(\rho) = \beta \gamma_{\rm E} \left[Q, [Q, \mathcal{U}_{\rm H}]_{\rho} \right]$, where $\mathcal{U}_{\rm H} = \mathcal{H} + \beta^{-1} \log \rho$ is the Helmholtz free energy operator [11]. According to the master equation (2), the time evolution of the Helmholtz free energy $\langle \mathcal{U}_{\rm H} \rangle = \operatorname{Tr}(\mathcal{U}_{\rm H}\rho)$ is governed by

$$\frac{\mathrm{d}\langle \mathcal{U}_{\mathrm{H}}\rangle}{\mathrm{d}t} = -\beta\gamma_{\mathrm{E}}\operatorname{Tr}\left(\mathcal{C}_{\rho}\mathcal{C}\right) , \qquad (4)$$

where $C = i [Q, U_{\rm H}]$, and thus $d \langle U_{\rm H} \rangle / dt \leq 0$ (since $C^{\dagger} = C$) [12, 16], i.e. the Helmholtz free energy $\langle U_{\rm H} \rangle$ is a monotonically decreasing function of time.

Note that the operator \mathcal{C} vanishes at the fixed point ρ_0 given by Eq. (1). Alternatively, the Kubo's identity [given by Eq. (4.2.17) of [16]] can be used to show that ρ_0 is a fixed point [11]. For some cases the existence of a limit cycle (i.e. periodic) solution for the GME (2) can be ruled out using Eq. (4). Along such a solution the condition $\mathcal{C} = 0$ must be satisfied [since Tr ($\mathcal{C}_{\rho}\mathcal{C}$) = 0 implies that $\mathcal{C} = 0$ when Tr $\rho^2 < 1$]. Hence, when $\rho = \rho_0$ is a unique solution of $\mathcal{C} = 0$, a limit cycle solution can be ruled out.

A linear master equation can be derived by replacing the nonlinear term $\Theta_{\rm B}(\rho)$ by the term $(\beta'/\hbar) \gamma_{\rm E}[Q, [Q, \mathcal{H}]]$, where $\beta' > 0$. It was shown in Ref. [17] (see also appendix B of Ref. [8]) that such a linear master equation is stable provided that $\gamma_{\rm E} > 0$. Below we analyze the stability of the nonlinear GME (2).

The stability of the fixed point ρ_0 of the master equation (2) is explored by the method of linearization applied to the nonlinear term $\Theta_{\rm B}(\rho)$. In the vicinity of $\rho_0 = {\rm diag}(\rho_1, \rho_2, \cdots, \rho_{d_{\rm H}})$ the density matrix ρ is expressed as $\rho = \rho_0 + \epsilon \mathcal{V}$, where ϵ is a real small parameter. Let $u\rho u^{\dagger} = \rho_{\rm d} = {\rm diag}(\rho'_1, \rho'_2, \cdots, \rho'_{d_{\rm H}})$ be diagonal, where u is unitary, i.e. $u^{\dagger}u = 1$. With the help of time-independent perturbation theory one finds that the eigenvalues ρ'_n of ρ are given by

$$\rho_n' = \rho_n + \epsilon \left(n | \mathcal{V} | n \right) + O\left(\epsilon^2 \right) , \qquad (5)$$

and the unitary transformation u that diagonalizes ρ is given by

$$u = \sum_{n} \left(|n| + \sum_{k \neq n} \frac{\epsilon \left(k | \mathcal{V} | n\right)}{\rho_n - \rho_k} | k \right) \left(n | + O\left(\epsilon^2\right) \right),$$
(6)

or

$$u = 1 - i\epsilon F + O\left(\epsilon^2\right) , \qquad (7)$$

where the Hermitian matrix F is given by

$$F = \sum_{k \neq l} \frac{i\left(k \mid \mathcal{V} \mid l\right)}{\rho_l - \rho_k} \mid k\right) \left(l \mid , \qquad (8)$$

 $(k | \mathcal{V} | l) = \mathcal{V}_{kl}$ is the (k'th raw - l'th column) matrix element of \mathcal{V} , and $|k\rangle$ (l| denotes a $d_{\mathrm{H}} \times d_{\mathrm{H}}$ matrix having entry 1 in the (k'th raw - l'th column), and entry 0 elsewhere.

Using the identity [12]

$$\int_{0}^{1} x^{\eta} y^{1-\eta} \mathrm{d}\eta = \mathcal{F}(x, y) , \qquad (9)$$

where

$$\mathcal{F}(x,y) = \frac{x-y}{\log x - \log y} , \qquad (10)$$

one finds that (recall that ρ_d is diagonal)

$$\int_0^1 \mathrm{d}\eta \; \rho_\mathrm{d}^\eta A \rho_\mathrm{d}^{1-\eta} = \mathcal{F}' \circ A \;, \tag{11}$$

where \circ denotes the Hadamard matrix multiplication (element by element matrix multiplication), and where the matrix elements of \mathcal{F}' are given by $\mathcal{F}'_{nm} = \mathcal{F}(\rho'_n, \rho'_m)$. Note that $\mathcal{F}'_{nm} = \mathcal{F}_{nm} + O(\epsilon)$, where $\mathcal{F}_{nm} = \mathcal{F}(\rho_n, \rho_m)$ [see Eq. (5)], hence, the following holds [see Eqs. (3) and (7) and note that $uAu^{\dagger} = A + i\epsilon [A, F] + O(\epsilon^2)$]

$$A_{\rho} = \mathcal{F}' \circ A + i\epsilon \left[F, \mathcal{F} \circ A\right] + i\epsilon \mathcal{F} \circ \left[A, F\right] + O\left(\epsilon^2\right) , \quad (12)$$

where $\mathcal{F}' = \mathcal{F} + \epsilon \left(\frac{d\mathcal{F}}{d\epsilon} \right) + O(\epsilon^2)$. The following holds [see Eq. (10)]

$$\mathcal{F}(x,y) = \frac{x+y}{2} f_{\rm D}\left(\frac{x-y}{x+y}\right) , \qquad (13)$$

where the function $f_{\rm D}(\eta)$ is given by

$$f_{\rm D}(\eta) = \frac{2\eta}{\log\frac{1+\eta}{1-\eta}} = \frac{\eta}{\tanh^{-1}\eta} \,. \tag{14}$$

The function $f_{\rm D}$ is symmetric, i.e. $f_{\rm D}(-\eta) = f_{\rm D}(\eta)$, and the following holds $f_{\rm D}(0) = 1$ and $f_{\rm D}(\pm 1) = 0$. With the help of Eqs. (5) and (13) one finds that the matrix $d\mathcal{F}/d\epsilon$ is real, symmetric, and the following holds (no summation due to repeated indices n and m)

$$\left(\frac{\mathrm{d}\mathcal{F}}{\mathrm{d}\epsilon}\right)_{nm} = \frac{\mathrm{d}\alpha_{nm}}{\mathrm{d}\epsilon}F_{nm} + \alpha_{nm}\frac{\mathrm{d}\eta_{nm}}{\mathrm{d}\epsilon}F'_{nm} ,\qquad(15)$$

where $\alpha_{nm} = (\rho_n + \rho_m)/2$, $\eta_{nm} = (\rho_n - \rho_m)/(\rho_n + \rho_m)$, $F_{nm} = f_D(\eta_{nm})$, and where $F'_{nm} = f'_D(\eta_{nm})$. Moreover, $\text{Tr}(d\mathcal{F}/d\epsilon) = 0$ (note that $F_{nn} = 1$ and $F'_{nn} = 0$). The $d_H^2 - 1$ Hermitian and trace-less $d_H \times d_H$ generalined Cell Mann metrics.) which even the $\text{SU}(d_{-})$ Lie

The $d_{\rm H}^2 - 1$ Hermitian and trace-less $d_{\rm H} \times d_{\rm H}$ generalized Gell-Mann matrices λ_n , which span the SU($d_{\rm H}$) Lie algebra, satisfy the orthogonality relation

$$\frac{\operatorname{Tr}\left(\lambda_a\lambda_b\right)}{2} = \delta_{ab} \ . \tag{16}$$

For the case $d_{\rm H} = 2$ ($d_{\rm H} = 3$) the matrices are called Pauli (Gell-Mann) matrices. The set { λ_a } of $d_{\rm H}^2 - 1$ matrices can be divided into three subsets. The subset { $\lambda_{{\rm X},(n,m)}$ } contains $d_{\rm H} (d_{\rm H} - 1)/2$ matrices given by $\lambda_{{\rm X},(n,m)} = |n\rangle (m| + |m\rangle (n|$, and the subset { $\lambda_{{\rm Y},(n,m)}$ } contains $d_{\rm H} (d_{\rm H} - 1)/2$ matrices given by $\lambda_{{\rm Y},(n,m)} =$ $-i |n\rangle (m| + i |m\rangle (n|$, where $1 \le m < n \le d_{\rm H}$. The subset { $\lambda_{{\rm Z},l}$ } contains $d_{\rm H} - 1$ diagonal matrices given by

$$\lambda_{Z,l} = \sqrt{\frac{2}{l(l+1)}} \left(-l|l+1) \left(l+1 \right) + \sum_{j=1}^{l} |j| \left(j \right) \right),$$
(17)

where $1 \leq l \leq d_{\rm H} - 1$.

It is convenient to express the perturbation $\epsilon \mathcal{V} = \rho - \rho_0$ as $\epsilon \mathcal{V} = \bar{\kappa} \cdot \bar{\lambda}$, where $\bar{\kappa} = \left(\kappa_1, \kappa_2, \cdots, \kappa_{d_{\mathrm{H}}^2-1}\right)$ and $\bar{\lambda} = \left(\lambda_1, \lambda_2, \cdots, \lambda_{d_{\mathrm{H}}^2-1}\right)$. In this notation the GME (2) becomes (repeated index implies summation)

$$\frac{\mathrm{d}\kappa_b}{\mathrm{d}t}\lambda_b = \Theta\left(\rho_0 + \kappa_b\lambda_b\right) \,,\tag{18}$$

or [see Eq. (16)]

$$\frac{\mathrm{d}\kappa_a}{\mathrm{d}t} = \frac{1}{2} \operatorname{Tr} \left(\Theta \left(\rho_0 + \kappa_b \lambda_b \right) \lambda_a \right) \ . \tag{19}$$

To first order in $\bar{\kappa}$

$$\frac{\mathrm{d}\kappa_a}{\mathrm{d}t} = \frac{1}{2} \operatorname{Tr} \left(\frac{\partial \Theta}{\partial \kappa_b} \lambda_a \kappa_b \right) , \qquad (20)$$

or in a vector form

$$\frac{\mathrm{d}\bar{\kappa}}{\mathrm{d}t} = J\bar{\kappa} , \qquad (21)$$

where the Jacobian matrix J is given by $J = J_{u} - J_{A} - J_{B}$, and where

$$J_{\Sigma} = \frac{1}{2} \operatorname{Tr} \left(\frac{\partial \Theta_{\Sigma}}{\partial \kappa_b} \lambda_a \right) , \qquad (22)$$

with $\Sigma \in \{u, A, B\}$.

The system's stability depends on the set of eigenvalues of the Jacobian matrix J, which is denoted by S. The system is stable provided that real (ξ) < 0 for any $\xi \in S$. It was shown in appendix B of Ref. [8] that such a system is stable provided that $J_{\rm u}$, $J_{\rm A}$ and $J_{\rm B}$ are all real, $J_{\rm u}$ is antisymmetric, all diagonal elements of $J_{\rm A} + J_{\rm B}$ are positive, and $d_{\rm H}$ is finite. Properties of the matrices $J_{\rm u}$, $J_{\rm A}$ and $J_{\rm B}$ are analyzed below.

The matrix J_u , which governs the unitary evolution, is given by [recall the trace identity $\operatorname{Tr}(XY) = \operatorname{Tr}(YX)$]

$$J_{\rm u} = \frac{i}{2\hbar} \operatorname{Tr} \left(\left[\lambda_b, \mathcal{H} \right] \lambda_a \right)$$
$$= \frac{i}{2\hbar} \operatorname{Tr} \left(\mathcal{H} \left[\lambda_a, \lambda_b \right] \right) , \qquad (23)$$

hence J_{u} is *real* and *antisymmetric* provided that \mathcal{H} is Hermitian (note that $i [\lambda_b, \lambda_a]$ is Hermitian).

The matrix $J_{\rm A}$ is given by

$$J_{A} = \frac{\gamma_{E}}{2} \operatorname{Tr} \left(\left[Q, \left[Q, \lambda_{b} \right] \right] \lambda_{a} \right)$$
$$= \frac{\gamma_{E}}{2} \operatorname{Tr} \left(- \left[Q, \lambda_{b} \right] \left[Q, \lambda_{a} \right] \right) .$$
(24)

Both matrices $i[Q, \lambda_a]$ and $i[Q, \lambda_b]$ are Hermitian, provided that Q is Hermitian, hence J_A is *real* (recall that γ_E is positive). The *diagonal* elements of J_A are *positive* since $-[Q, \lambda_b][Q, \lambda_a]$ is positive-definite for the case a = b.

The diagonal elements of the matrix $J_{\rm B}$ cab be evaluated using the linearization of the term A_{ρ} given by Eq. (12). For the case where the perturbation $\mathcal{V} = (\rho - \rho_0) / \epsilon$ is a generalized Gell-Mann matrix, i.e. $\mathcal{V} \in \{\lambda_a\}$, the following holds [see Eq. (8)]

$$F = \begin{cases} \frac{\lambda_{Y,(n,m)}}{\rho_n - \rho_m} & \text{if } \mathcal{V} = \lambda_{X,(n,m)} \\ -\frac{\lambda_{X,(n,m)}}{\rho_n - \rho_m} & \text{if } \mathcal{V} = \lambda_{Y,(n,m)} \end{cases},$$
(25)

and [see Eq. (12), and note that, according to Eq. (5), $\mathcal{F}' = \mathcal{F} + O(\epsilon^2)$ when all diagonal elements of the perturbation vanish, e.g. when $\mathcal{V} \in \{\lambda_{X,(n,m)}\} \cup \{\lambda_{Y,(n,m)}\}$, and, according to Eqs. (7) and (8), $u = 1 + O(\epsilon^2)$ when the perturbation is diagonal, e.g. when $\mathcal{V} \in \{\lambda_{Z,l}\}$]

$$\frac{\mathrm{d}A_{\rho}}{\mathrm{d}\epsilon} = \begin{cases} \frac{\left[\mathcal{F} \circ A, \lambda_{\mathrm{Y},(n,m)}\right] - \mathcal{F} \circ \left[A, \lambda_{\mathrm{Y},(n,m)}\right]}{i(\rho_{n} - \rho_{m})} & \text{if } \mathcal{V} = \lambda_{\mathrm{X},(n,m)} \\ \frac{\left[\mathcal{F} \circ A, \lambda_{\mathrm{X},(n,m)}\right] - \mathcal{F} \circ \left[A, \lambda_{\mathrm{X},(n,m)}\right]}{(-i)(\rho_{n} - \rho_{m})} & \text{if } \mathcal{V} = \lambda_{\mathrm{Y},(n,m)} \\ \frac{\mathrm{d}\mathcal{F}'}{\mathrm{d}\epsilon} \circ A & \text{if } \mathcal{V} = \lambda_{\mathrm{Z},(n,m)} \end{cases}$$
(26)

The diagonal elements of $J_{\rm A} + J_{\rm B}$ are evaluated by using of Eq. (26) with different values of the perturbation \mathcal{V} .

The diagonal matrix element corresponding to the generalized Gell-Mann matrix $\lambda_{Z,l}$, which is labeled by j_l , is given by [see Eqs. (22), (24) and (26)]

$$j_{l} = \frac{\gamma_{\rm E}}{2} \operatorname{Tr} \left(- [Q, \lambda_{\rm Z,l}]^{2} \right) + \frac{\beta \gamma_{\rm E}}{2} \operatorname{Tr} \left(\left[Q, \frac{\mathrm{d}\mathcal{F}}{\mathrm{d}\epsilon} \circ [Q, \mathcal{H}] \right] \lambda_{\rm Z,l} \right) , \qquad (27)$$

where the term $d\mathcal{F}/d\epsilon$ is evaluated according to Eq. (15) for the case where the perturbation is given by $\mathcal{V} = \lambda_{Z,l}$. In terms of the elements of the diagonal matrix $\lambda_{Z,l} =$ $\operatorname{diag}(\nu_1, \nu_2, \cdots, \nu_{d_{\mathrm{H}}})$ one finds using Eq. (5) that $\rho'_n =$ $\rho_n + \epsilon \nu_n + O(\epsilon^2)$, hence $(d\mathcal{F}/d\epsilon)_{nm} = d_{nm}$, where

$$d_{nm} = \frac{\nu_{nm}F_{nm}}{2\varkappa_{nm}} \left(1 + \frac{(\varkappa_{nm} - \eta_{nm})F'_{nm}}{F_{nm}}\right) , \qquad (28)$$

 $\nu_{nm} = \nu_n - \nu_m$ and $\varkappa_{nm} = (\nu_n - \nu_m) / (\nu_n + \nu_m)$. The following holds $d_{nm} = d_{mn}$, hence Eq. (27) yields

$$j_l = \gamma_{\rm E} \sum_{n < m} \zeta_{nm} \nu_{nm}^2 \left| q_{nm} \right|^2 \,, \tag{29}$$

where $\zeta_{nm} = 1 + d_{nm}e_{nm}/\nu_{nm}$, $e_{nm} = \beta (E_n - E_m)$, and where q_{nm} are the matrix elements of the operator Q(recall that it is assumed that $Q^{\dagger} = Q$, i.e. $q_{mn} = q_{nm}^*$). With the help of the relation $\eta_{nm} = -\tanh(e_{nm}/2)$ [see Eq. (1)] one finds that $\zeta_{nm} = \zeta(\eta_{nm}, \varkappa_{nm})$, where the function $\zeta(\eta, \varkappa)$ is given by [see Eq. (14) and note that $1 - (1/(1 - \eta^2)) (\eta/\tanh^{-1}\eta) = \eta F'(\eta)/F(\eta)$]

$$\zeta(\eta, \varkappa) = \frac{f_{\rm D}(\eta)}{1 - \eta^2} \left(1 - \frac{\eta}{\varkappa}\right) \,. \tag{30}$$

The following holds [see Eq. (17), and note that only the cases for which $v_{nm} \neq 0$, i.e. the cases that can contribute to j_l , are listed]

$$-\frac{1}{\varkappa_{nm}} = \begin{cases} \frac{l-1}{l+1} & n \le l \text{ and } m = l+1\\ 1 & n \le l \text{ and } m > l+1\\ 1 & n = l+1 \text{ and } m > l+1 \end{cases}$$
(31)

hence $0 \leq (-1/\varkappa) \leq 1$ for all terms contributing to j_l , hence $\zeta_{nm}\nu_{nm}^2 \geq 0$ for these terms, and consequently $j_l \geq 0$.

The diagonal matrix element corresponding to the generalized Gell-Mann matrix $\lambda_{X,(2,1)}$ ($\lambda_{Y,(2,1)}$) is labelled by j_X (j_Y). We show below that both j_X and j_Y are nonnegative. The proof is applicable for all other diagonal elements, corresponding to all generalized Gell-Mann matrices $\lambda \in \{\lambda_{X,(n,m)}\} \cup \{\lambda_{Y,(n,m)}\}$ with $(n,m) \neq (2,1)$, since the ordering of the energy eigenvectors is arbitrary.

With the help of Eqs. (22), (24) and (26) one finds that [the subscript (2, 1) is omitted for brevity]

$$\frac{j_{\mathrm{X}}}{\frac{\gamma_{\mathrm{E}}}{2}} = \mathrm{Tr}\left(-\left[Q,\lambda_{\mathrm{X}}\right]\left[Q,\lambda_{\mathrm{X}}\right]\right) + \mathrm{Tr}\left(\beta\left[Q,\frac{\left[\mathcal{F}\circ\left[Q,\mathcal{H}\right],\lambda_{\mathrm{Y}}\right]-\mathcal{F}\circ\left[\left[Q,\mathcal{H}\right],\lambda_{\mathrm{Y}}\right]}{i\left(\rho_{2}-\rho_{1}\right)}\right]\lambda_{\mathrm{X}}\right),$$
(32)

and

$$\frac{\hat{J}Y}{\frac{\gamma_{\rm E}}{2}} = \operatorname{Tr}\left(-\left[Q, \lambda_{\rm Y}\right]\left[Q, \lambda_{\rm Y}\right]\right) \\
+ \operatorname{Tr}\left(\beta \left[Q, \frac{\left[\mathcal{F} \circ \left[Q, \mathcal{H}\right], \lambda_{\rm X}\right] - \mathcal{F} \circ \left[\left[Q, \mathcal{H}\right], \lambda_{\rm X}\right]}{\left(-i\right)\left(\rho_2 - \rho_1\right)}\right] \lambda_{\rm Y}\right) \\$$
(33)

hence

$$\frac{j_{\rm X}}{\gamma_{\rm E}} = q_{\rm d}^2 + 4\upsilon q_{12}^{\prime\prime 2} + \sum_{n=1}^2 \sum_{m\geq 3} G_{nm} \left| q_{nm} \right|^2 , \qquad (34)$$

and

$$\frac{j_{\rm Y}}{\gamma_{\rm E}} = q_{\rm d}^2 + 4\upsilon q_{12}^{\prime 2} + \sum_{n=1}^2 \sum_{m \ge 3} G_{nm} \left| q_{nm} \right|^2 , \qquad (35)$$

where $q_{\rm d} = q_{11} - q_{22}$,

$$\upsilon = 1 - \frac{\left(\mathcal{F}_{11} + \mathcal{F}_{22} - 2\mathcal{F}_{12}\right)e_{12}}{2\left(\rho_1 - \rho_2\right)}, \qquad (36)$$

 $q'_{12} = \operatorname{Re} q_{12}, q''_{12} = \operatorname{Im} q_{12}, \text{ and where}$

$$G_{nm} = 1 + \frac{(\mathcal{F}_{1m} - \mathcal{F}_{2m}) e_{nm}}{\rho_1 - \rho_2} .$$
 (37)

With the help of Eqs. (1), (13) and (14) one finds that [note that $e_{nm} = -\log(\rho_n/\rho_m) = \log((1 - \eta_{nm})/(1 + \eta_{nm})) = -2\eta_{nm}/f_D(\eta_{nm})$]

$$\upsilon = \frac{1}{f_{\rm D}\left(\eta_{12}\right)} \,, \tag{38}$$

and that $G_{1m} = G(\rho_1/\rho_m, \rho_2/\rho_m)$ and $G_{2m} = G(\rho_2/\rho_m, \rho_1/\rho_m)$, where the function G is given by

$$G(r_1, r_2) = 1 - \frac{\frac{r_1 - 1}{\log r_1} - \frac{r_2 - 1}{\log r_2}}{r_1 - r_2} \log r_1 , \qquad (39)$$

or

$$G(r_1, r_2) = \frac{r_2 - 1}{r_2 \log r_2} \frac{\log \frac{r_1}{r_2}}{\frac{r_1}{r_2} - 1}, \qquad (40)$$

hence $v \geq 1$ [since $0 \leq f_{\rm D}(\eta_{12}) \leq 1$] and $G_{nm} \geq 0$ [see Eq. (40), and note that for non-negative r_1 and r_2 , both the first factor, which depends on r_2 only, and the second one, which depends on r_1/r_2 only, are non-negative], and thus both $j_{\rm X}$ and $j_{\rm Y}$ are non-negative.

In summary, the dynamics governed by the GME (2) in the vicinity of the steady state ρ_0 depends on the $d_{\rm H}^2 - 1$ diagonal element of the Jacobean matrix $J_{\rm A} + J_{\rm B}$. Our derived expressions for the eigenvalues, given by Eqs. (29), (34) and (35), can be used to evaluate statistical properties of the system near its steady state ρ_0 . We find that all these eigenvalues are non-negative, and conclude that the steady state ρ_0 is stable. This raises the question under what conditions dynamical instability is possible in a quantum Hilbert space of finite dimensionality.

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