Stability of the Grabert master equation

Eyal Buks and Dvir Schwartz
Andrew and Erna Viterbi Department of Electrical Engineering, Technion, Haifa 32000, Israel
(Dated: March 17, 2021)

We study the dynamics of a quantum system having Hilbert space of finite dimension $d_H$. Instabilities are possible provided that the master equation governing the system’s dynamics contain nonlinear terms. Here we consider the nonlinear master equation derived by Grabert. The dynamics near a fixed point is analyzed by using the method of linearization, and by evaluating the eigenvalues of the Jacobian matrix. We find that all these eigenvalues are non-negative, and conclude that the fixed point is stable. This finding raises the question: under what conditions instability is possible in a quantum system having finite $d_H$?

PACS numbers:

Consider a given closed quantum system having Hilbert space of finite dimension $d_H$, whose master equation, which governs the time evolution of the reduced density matrix $\rho$, can be expressed as $d\rho/dt = \Theta(\rho) = \Theta_u(\rho) - \Theta_d(\rho)$. The first term, which is given by $\Theta_u(\rho) = (i/\hbar) [\rho, \hat{H}]$, where $\hat{H} = \hat{H}^I$ is the Hamiltonian of the closed system, represents unitary evolution, and the second one $\Theta_d(\rho)$ represents the effect of coupling between the closed system and its environment. While it is commonly assumed that both the unitary term $\Theta_u(\rho)$ and the damping term $\Theta_d(\rho)$ are linear in $\rho$, in some cases the master equation can become nonlinear. Two types of nonlinearity are considered below.

For the first one, which is henceforth referred to as unitary nonlinearity, the unitary term $\Theta_u(\rho)$ is replaced by a nonlinear term. In most cases, unitary nonlinearity originates from either the mean field approximation or from a transformation mapping the Hilbert space of finite dimension $d_H$ into a space having infinite dimensionality (e.g., the Holstein-Primakoff transformation, which can yield a parametric instability in ferromagnetic resonators). Here, we consider the second type, which is henceforth referred to as damping nonlinearity, and focus on the master equation that was proposed by Grabert, which has a damping term $\Theta_d(\rho)$ nonlinear in $\rho$.

Grabert has shown that the invalidity of the quantum regression hypothesis gives rise to damping nonlinearity. The nonlinear term added to the master equation ensures that the purity $\text{Tr}\rho^2$ does not exceed unity and that entropy is generated at a non-negative rate, as is expected from the second law of thermodynamics. Note, however, that under appropriate conditions, nonlinear dynamics may allow for faster than light signaling.

The Grabert master equation (GME) has a fixed point given by

$$\rho_0 = \frac{e^{-\beta \hat{H}}}{\text{Tr} (e^{-\beta \hat{H}})},$$

where $\beta = 1/(k_B T)$ is the inverse of the thermal energy. At the fixed point $\rho_0$ the system is in thermal equilibrium having Boltzmann distribution.

Here we explore the stability of this fixed point $\rho_0$ for the case where the Hamiltonian $\hat{H}$ of the closed system is time-independent. In a basis of energy eigenstates of a time-independent Hamiltonian both matrices $\hat{H} = \text{diag} (E_1, E_2, \cdots, E_{d_H})$ and $\rho_0 = \text{diag} (\rho_1, \rho_2, \cdots, \rho_{d_H})$ are diagonal, where $\rho_n = e^{\beta E_n}/\text{Tr} (e^{-\beta \hat{H}})$ [see Eq. (1)].

For the case of thermal equilibrium, one may argue that the stability of $\rho_0$ is obvious. However, the stability of a driven system is anything but obvious. Note that in many cases the rotating wave approximation (RWA) is employed in order to model the dynamics of a given system under external driving, using a transformation into a rotating frame, in which the Hamiltonian becomes time-independent in the RWA. Thus, our conclusion, that the fixed point $\rho_0$ is stable for any time independent Hermitian $\hat{H}$, can be extended beyond the limits of thermal equilibrium.

The GME for the reduced density matrix $\rho$ can be expressed as

$$\frac{d\rho}{dt} = \Theta(\rho) = \Theta_u(\rho) - \Theta_d(\rho),$$

where the damping term is given by $\Theta_d(\rho) = \Theta_A(\rho) + \Theta_B(\rho)$, where $\Theta_A(\rho)$, which is given by $\Theta_A(\rho) = \gamma_E [Q, [Q, \rho]]$, is linear in $\rho$, and $\Theta_B(\rho)$, which is given by $\Theta_B(\rho) = \beta \gamma_E [Q, Q, \rho]$ is nonlinear. The constant $\gamma_E > 0$ is a damping rate, the Hermitian operator $Q = \hat{Q}$ describes the interaction between the quantum system and its environment, and

$$A_\rho = \int_0^1 d\eta \rho^n A_\rho^{1-n}.$$ (3)

Alternatively, the damping term $\Theta_d(\rho)$ can be expressed as $\Theta_d(\rho) = \beta \gamma_E [Q, \hat{U}_H]_\rho$, where $\hat{U}_H = \hat{H} + \beta^{-1} \log \rho$ is the Helmholtz free energy operator. According to the master equation, the time evolution of the Helmholtz free energy $\langle \hat{U}_H \rangle = \text{Tr} (\hat{U}_H \rho)$ is governed by

$$\frac{d \langle \hat{U}_H \rangle}{dt} = -\beta \gamma_E \text{Tr} (C_\rho C),$$ (4)
where $C = i [Q, U_H]$, and thus $d \langle U_H \rangle / dt \leq 0$ (since $C^\dagger = C$) [12, 14], i.e. the Helmholtz free energy $\langle U_H \rangle$ is a monotonically decreasing function of time.

Note that the operator $C$ vanishes at the fixed point $\rho_0$ given by Eq. (1). Alternatively, the Kubo's identity [given by Eq. (4.2.17) of [16]] can be used to show that $\rho_0$ is a fixed point [11]. For some cases the existence of a limit cycle (i.e. periodic) solution for the GME [2] can be ruled out using Eq. (4). Along such a solution the condition $C = 0$ must be satisfied [since $\text{Tr} (C \rho C) = 0$ implies that $C = 0$ when $\text{Tr} \rho^2 < 1$]. Hence, when $\rho = \rho_0$ is a unique solution of $C = 0$, a limit cycle solution can be ruled out.

A linear master equation can be derived by replacing the nonlinear term $\Theta (\rho)$ by the term $(\beta^\dagger / \hbar) \gamma_E [Q, [Q, \mathcal{H}]]$, where $\beta^\dagger > 0$. It was shown in Ref. [17] (see also appendix B of Ref. [8]) that such a linear master equation is stable provided that $\gamma_E > 0$. Below we analyze the stability of the nonlinear GME [2].

The stability of the fixed point $\rho_0$ of the master equation [2] is explored by the method of linearization applied to the nonlinear term $\Theta (\rho)$. In the vicinity of $\rho_0 = \text{diag} (\rho_1, \rho_2, \ldots, \rho_{d_H})$ the density matrix $\rho$ is expressed as $\rho = \rho_0 + \epsilon \mathcal{V}$, where $\epsilon$ is a real small parameter. Let $u \rho u^\dagger = \rho_0 = \text{diag} (\rho_1', \rho_2', \ldots, \rho_{d_H}')$ be diagonal, where $u$ is unitary, i.e. $u^\dagger u = 1$. With the help of time-independent perturbation theory one finds that the eigenvalues $\rho_n'$ of $\rho$ are given by

$$\rho_n' = \rho_n + \epsilon (n | \mathcal{V} | n) + O (\epsilon^2),$$

and the unitary transformation $u$ that diagonalizes $\rho$ is given by

$$u = \sum_n \left( | n \rangle + \sum_{k \neq n} \frac{\epsilon (k | \mathcal{V} | n)}{\rho_n - \rho_k} | k \rangle \right) (n | + O (\epsilon^2),$$

or

$$u = 1 - i \epsilon F + O (\epsilon^2),$$

where the Hermitian matrix $F$ is given by

$$F = \sum_{k \neq l} \frac{i (k | \mathcal{V} | l)}{\rho_l - \rho_l} | k \rangle \langle l |,$$

$(k | \mathcal{V} | l) = \mathcal{V}_{kl}$ is the $(k \text{'th raw } - l \text{'th column})$ matrix element of $\mathcal{V}$, and $| k \rangle \langle l |$ denotes a $d_H \times d_H$ matrix having entry 1 in the $(k \text{'th raw } - l \text{'th column})$, and entry 0 elsewhere.

Using the identity [12]

$$\int_0^1 x^n y^{1-n} d\eta = F (x, y),$$

where

$$F (x, y) = \frac{x - y}{\log x - \log y},$$

one finds that (recall that $\rho_0$ is diagonal)

$$\int_0^1 d\eta \rho_0^n A \rho_0^{1-n} = F' \circ A,$$

where $\circ$ denotes the Hadamard matrix multiplication (element by element matrix multiplication), and where the matrix elements of $F'$ are given by $F'_{nm} = F (\rho_n', \rho_m')$. Note that $F'_{nm} = F_{nm} + O (\epsilon)$, where $F_{nm} = F (\rho_n, \rho_m)$ [see Eq. (5)], hence, the following holds [see Eqs. (5) and (7)] and note that $u A u^\dagger = A + i \epsilon [A, F] + O (\epsilon^2)$

$$A_\rho = F' \circ A + i \epsilon [F, F \circ A] + i \epsilon F \circ [A, F] + O (\epsilon^2),$$

where $F' = F + \epsilon (dF/d\epsilon) + O (\epsilon^2)$.

The following holds [see Eq. (10)]

$$F (x, y) = \frac{x + y}{2} f_D \left( \frac{x - y}{x + y} \right),$$

where the function $f_D (\eta)$ is given by

$$f_D (\eta) = \frac{2\eta}{\log 1/\eta} = \eta \tan^{-1} \eta.$$

The function $f_D$ is symmetric, i.e. $f_D (-\eta) = f_D (\eta)$, and the following holds $f_D (0) = 1$ and $f_D (\pm 1) = 0$. With the help of Eqs. (5) and (13) one finds that the matrix $dF/d\epsilon$ is real, symmetric, and the following holds (no summation due to repeated indices $n$ and $m$)

$$\left( \frac{dF}{d\epsilon} \right)_{nm} = \frac{d\alpha_{nm}}{d\epsilon} F_{nm} + \alpha_{nm} \frac{d\eta_{nm}}{d\epsilon} F'_{nm},$$

where $\alpha_{nm} = (\rho_n + \rho_m) / 2$, $\eta_{nm} = (\rho_n - \rho_m) / (\rho_n + \rho_m)$, $F_{nm} = F_D (\eta_{nm})$, and where $F'_{nm} = f_D' (\eta_{nm})$. Moreover, $\text{Tr} (dF/d\epsilon) = 0$ (note that $F_{nn} = 1$ and $F'_{nn} = 0$).

The $d_H^2 - 1$ Hermitian and trace-less $d_H \times d_H$ generalized Gell-Mann matrices $\lambda_{\alpha}$, which span the $\text{SU}(d_H)$ Lie algebra, satisfy the orthogonality relation

$$\frac{\text{Tr} (\lambda_{\alpha} \lambda_{\beta})}{2} = \delta_{ab}.$$

For the case $d_H = 2$ ($d_H = 3$) the matrices are called Pauli (Gell-Mann) matrices. The set $\{ \lambda_{\alpha} \}$ of $d_H^2 - 1$ matrices can be divided into three subsets. The subset $\{ \lambda_{\chi_{(n,m)}} \}$ contains $d_H (d_H - 1)/2$ matrices given by $\lambda_{\chi_{(n,m)}} = | n \rangle \langle m | + | m \rangle \langle n |$, and the subset $\{ \lambda_{\chi_{(n,m)}} \}$ contains $d_H (d_H - 1)/2$ matrices given by $\lambda_{\chi_{(n,m)}} = -i | n \rangle \langle m | + i | m \rangle \langle n |$, where $1 \leq m < n \leq d_H$. The subset $\{ \lambda_{\chi_{l}} \}$ contains $d_H - 1$ diagonal matrices given by

$$\lambda_{\chi_{l}} = \sqrt{\frac{2}{l (l + 1)}} \left( -l | l + 1 \rangle \langle l + 1 | + \sum_{j=1}^{l} | j \rangle \langle j | \right),$$

where $1 \leq l \leq d_H - 1$. 


It is convenient to express the perturbation $\epsilon \mathcal{V} = \rho - \rho_0$ as $\epsilon \mathcal{V} = \mathring{\kappa} \cdot \mathring{\lambda}$, where $\mathring{\kappa} = (\kappa_1, \kappa_2, \cdots, \kappa_{d_H-1})$ and $\mathring{\lambda} = (\lambda_1, \lambda_2, \cdots, \lambda_{d_H-1})$. In this notation the GME becomes (repeated index implies summation)

$$\frac{d\kappa_i}{dt} \lambda_b = \Theta (\rho_0 + \kappa_i \lambda_b) , \quad (18)$$

or [see Eq. (16)]

$$\frac{d\kappa_i}{dt} = -\frac{1}{2} \text{Tr} \left( \Theta (\rho_0 + \kappa_i \lambda_b) \right) \lambda_b . \quad (19)$$

To first order in $\mathring{\kappa}$

$$\frac{d\kappa_i}{dt} = \frac{1}{2} \text{Tr} \left( \frac{\partial \Theta}{\partial \kappa_i} \lambda_a \kappa_b \right) , \quad (20)$$

or in a vector form

$$\frac{d\mathring{\kappa}}{dt} = J\mathring{\kappa} , \quad (21)$$

where the Jacobian matrix $J$ is given by $J = J_a - J_A - J_B$, and where

$$J_\Sigma = \frac{1}{2} \text{Tr} \left( \frac{\partial \Theta_\Sigma}{\partial \kappa_b} \lambda_a \right) , \quad (22)$$

with $\Sigma \in \{ u, A, B \}$.

The system's stability depends on the set of eigenvalues of the Jacobian matrix $J$, which is denoted by $\mathcal{S}$. The system is stable provided that real $\xi < 0$ for any $\xi \in \mathcal{S}$. It was shown in appendix B of Ref. [5] that such a system is stable provided that $J_u, J_A$ and $J_B$ are all real, $J_u$ is antisymmetric, all diagonal elements of $J_A + J_B$ are positive, and $d_H$ is finite. Properties of the matrices $J_u$, $J_A$ and $J_B$ are analyzed below.

The matrix $J_u$, which governs the unitary evolution, is given by [recall the trace identity $\text{Tr} \left( X Y \right) = \text{Tr} \left( Y X \right)$]

$$J_u = \frac{i}{2} \text{Tr} \left( \left[ \lambda_b, \mathcal{H} \right] \lambda_a \right) \nonumber$$

$$= \frac{i}{2} \text{Tr} \left( \mathcal{H} \left[ \lambda_a, \lambda_b \right] \right) , \quad (23)$$

hence $J_u$ is real and antisymmetric provided that $\mathcal{H}$ is Hermitian (note that $i \left[ \lambda_b, \lambda_a \right]$ is Hermitian).

The matrix $J_A$ is given by

$$J_A = \frac{\gamma E}{2} \text{Tr} \left( \left[ Q, \left[ Q, \lambda_b \right] \right] \lambda_a \right) \nonumber$$

$$= \frac{\gamma E}{2} \text{Tr} \left( -\left[ Q, \lambda_a \right] \left[ Q, \lambda_a \right] \right) . \quad (24)$$

Both matrices $i \left[ Q, \lambda_b \right]$ and $i \left[ Q, \lambda_a \right]$ are Hermitian, provided that $Q$ is Hermitian, hence $J_A$ is real (recall that $\gamma E$ is positive). The diagonal elements of $J_A$ are positive since $-\left[ Q, \lambda_b \right] \left[ Q, \lambda_a \right]$ is positive-definite for the case $a = b$.

The diagonal elements of the matrix $J_B$ can be evaluated using the linearization of the term $A_u$ given by Eq. (12). For the case where the perturbation $\mathcal{V} = (\rho - \rho_0)/\epsilon$ is a generalized Gell-Mann matrix, $\mathcal{V} \in \{ \lambda_a \}$, the following holds [see Eq. (8)]

$$F = \begin{cases} \lambda_{Y_{V,(u,m)}} - \rho_{\nu_{(u,m)}} & \text{if } \mathcal{V} = \lambda_{X_{V,(u,m)}} \\ \rho_{\nu_{(u,m)}} & \text{if } \mathcal{V} = \lambda_{Y_{V,(u,m)}} \end{cases} \quad (25)$$

and [see Eq. (12), and note that, according to Eq. (5), $F' = F + O (\epsilon^2)$] when all diagonal elements of the perturbation vanish, e.g. when $\mathcal{V} \in \{ \lambda_{X_{V,(u,m)}} \} \cup \{ \lambda_{Y_{V,(u,m)}} \}$, and, according to Eqs. (7) and (8), $u = 1 + O (\epsilon^2)$ when the perturbation is diagonal, e.g. when $\mathcal{V} \in \{ \lambda_{X_{V,(u,m)}} \}$

$$\frac{dA_i}{d\epsilon} = \begin{cases} \frac{[F \circ A, \lambda_{Y_{V,(u,m)}}]}{\lambda_{X_{V,(u,m)}} - \rho_{\nu_{(u,m)}}} - \frac{[F \circ A, \lambda_{X_{V,(u,m)}}]}{\lambda_{Y_{V,(u,m)}} - \rho_{\nu_{(u,m)}}} & \text{if } \mathcal{V} = \lambda_{X_{V,(u,m)}} \\ \frac{[F \circ A, \lambda_{Y_{V,(u,m)}}]}{\lambda_{X_{V,(u,m)}} - \rho_{\nu_{(u,m)}}} - \frac{[F \circ A, \lambda_{X_{V,(u,m)}}]}{\lambda_{Y_{V,(u,m)}} - \rho_{\nu_{(u,m)}}} & \text{if } \mathcal{V} = \lambda_{Y_{V,(u,m)}} \end{cases} \quad (26)$$

The diagonal elements of $J_A + J_B$ are evaluated by using of Eq. (20) with different values of the perturbation $\mathcal{V}$.

The diagonal matrix element corresponding to the generalized Gell-Mann matrix $\lambda_{Z,t}$, which is labeled by $j_t$, is given by [see Eqs. (22), (21) and (26)]

$$j_t = \frac{\gamma E}{2} \text{Tr} \left( -[Q, \lambda_{Z,t}]^2 \right) + \frac{\beta \gamma E}{2} \text{Tr} \left( \left[ Q, \frac{dF}{d\epsilon} \circ \left[ Q, \mathcal{H} \right] \right] \lambda_{Z,t} \right) , \quad (27)$$

where the term $dF/d\epsilon$ is evaluated according to Eq. (15) where the perturbation is given by $\mathcal{V} = \lambda_{Z,t}$.

In terms of the elements of the diagonal matrix $\lambda_{Z,t} = \text{diag} \left( \nu_1, \nu_2, \cdots, \nu_{d_H} \right)$ one finds using Eq. (5) that $\rho_n = \rho_n + \epsilon \nu_n + O (\epsilon^2)$, hence $(dF/d\epsilon)_{nm} = d_{nm}$, where

$$d_{nm} = \frac{\nu_{nm} F_{nm}}{2 \xi_{nm}} \left( 1 + \frac{(\xi_{nm} - \nu_{nm}) F'_{nm}}{F_{nm}} \right) , \quad (28)$$

$$\nu_{nm} = \nu_n - \nu_m$$

and $\xi_{nm} = (\nu_n - \nu_m) / (\nu_n + \nu_m)$. The following holds $d_{nm} = d_{mn}$, hence Eq. (27) yields

$$j_t = \frac{\gamma E}{2} \sum_{n < m} \xi_{nm} \nu_n^2 \left| q_{nm} \right|^2 , \quad (29)$$

where $\xi_{nm} = 1 + d_{nm} \epsilon_{nm}/\nu_{nm}$, $\epsilon_{nm} = \beta (E_n - E_m)$, and where $q_{nm}$ are the matrix elements of the operator $Q$ (recall that it is assumed that $Q^2 = Q$, i.e. $q_{nm} = q_{nm}^*$). With the help of the relation $\eta_{nm} = -\tanh (\epsilon_{nm}/2)$ [see Eq. (11)] one finds that $\xi_{nm} = \zeta (\eta_{nm}, \xi_{nm})$, where the function $\zeta (\eta, \xi)$ is given by [see Eq. (13) and note that $1 - (1/ (1 - \eta^2)) (\eta / \tanh^{-1} \eta) = \eta F' (\eta) / F (\eta)$]

$$\zeta (\eta, \xi) = \frac{f_D (\eta)}{1 - \eta^2} \left( \frac{1 - \eta}{\xi} \right) . \quad (30)$$
The following holds [see Eq. (17), and note that only the cases for which \( v_{nm} \neq 0 \), i.e. the cases that can contribute to \( j_l \), are listed]

\[
- \frac{1}{\kappa} = \begin{cases} 
\frac{1}{t+1} & n \leq l \text{ and } m = l + 1 \\
1 & n \leq l \text{ and } m > l + 1 \\
1 & n = l + 1 \text{ and } m > l + 1 
\end{cases},
\]

hence \( 0 \leq (-1/\kappa) \leq 1 \) for all terms contributing to \( j_l \), hence \( \zeta nm \nu_{nm} ^2 \geq 0 \) for these terms, and consequently \( j_l \geq 0 \).

The diagonal matrix element corresponding to the generalized Gell-Mann matrix \( \lambda_{X,(2,1)} \) \( \lambda_{Y,(2,1)} \) is labelled by \( j_X \) \( j_Y \). We show below that both \( j_X \) and \( j_Y \) are non-negative. The proof is applicable for all other diagonal elements, corresponding to all generalized Gell-Mann matrices \( \lambda \in \{ \lambda_{X,(n,m)} \} \cup \{ \lambda_{Y,(n,m)} \} \) with \((n,m) \neq (2,1)\), since the ordering of the energy eigenvectors is arbitrary.

With the help of Eqs. (22), (24) and (26) one finds that the subscript \((2,1)\) is omitted for brevity

\[
\frac{j_X}{\gamma E} = \text{Tr} \left( -[Q, \lambda_X ] [Q, \lambda_X ] \right) \\
+ \text{Tr} \left( \beta \left[ Q, \left[ \mathcal{F} \circ [Q, \mathcal{H}], \lambda_X \right] - \mathcal{F} \circ [[Q, \mathcal{H}], \lambda_X ] \right] \right),
\]

and

\[
\frac{j_Y}{\gamma E} = \text{Tr} \left( -[Q, \lambda_Y ] [Q, \lambda_Y ] \right) \\
+ \text{Tr} \left( \beta \left[ Q, \left[ \mathcal{F} \circ [Q, \mathcal{H}], \lambda_X \right] - \mathcal{F} \circ [[Q, \mathcal{H}], \lambda_X ] \right] \right),
\]

hence

\[
\frac{j_X}{\gamma E} = q_d^2 + 4v q_{12}^2 + \sum_{n=1}^{2} \sum_{m \geq 3} G_{nm} |q_{nm}|^2,
\]

and

\[
\frac{j_Y}{\gamma E} = q_d^2 + 4v q_{12}^2 + \sum_{n=1}^{2} \sum_{m \geq 3} G_{nm} |q_{nm}|^2,
\]

where \( q_d = q_{11} - q_{22} \),

\[
v = 1 - \frac{(\mathcal{F}_{11} + \mathcal{F}_{22} - 2\mathcal{F}_{12}) e_{12}}{2(\rho_1 - \rho_2)},
\]

\[
G_{nm} = 1 + \frac{(\mathcal{F}_{1m} - \mathcal{F}_{2m}) e_{nm}}{\rho_1 - \rho_2}.
\]

With the help of Eqs. (11), (13) and (14) one finds that \( \eta_{nm} = -\log(\rho_n/\rho_m) = \log ((1 - \eta_{nm})/(1 + \eta_{nm})) = -2\eta_{nm}/f_D (\eta_{nm}) \)

\[
v = \frac{1}{f_D (\eta_{12})},
\]

and that \( G_1 = G(\rho_1/\rho_m, \rho_2/\rho_m) \) and \( G_2 = G(\rho_2/\rho_m, \rho_1/\rho_m) \), where the function \( G \) is given by

\[
G (r_1, r_2) = 1 - \frac{r_{1-1} - r_{2-1}}{r_1 - r_2} \log r_1,
\]

or

\[
G (r_1, r_2) = \frac{r_2 - 1}{r_2 \log r_2} \frac{\log r_2}{\log r_2 + 1},
\]

hence \( v \geq 1 \) [since \( 0 \leq f_D (\eta_{12}) \leq 1 \) and \( G_{nm} \geq 0 \) [see Eq. (40)], and note that for non-negative \( r_1 \) and \( r_2 \), both the first factor, which depends on \( r_2 \) only, and the second one, which depends on \( r_1/r_2 \) only, are non-negative], and thus both \( j_X \) and \( j_Y \) are non-negative.

In summary, the dynamics governed by the GME (2) in the vicinity of the steady state \( \rho_0 \) depends on the \( d_{11} - 1 \) diagonal element of the Jacobian matrix \( J_A + J_B \). Our derived expressions for the eigenvalues, given by Eqs. (29), (31) and (35), can be used to evaluate statistical properties of the system near its steady state \( \rho_0 \). We find that all these eigenvalues are non-negative, and conclude that the steady state \( \rho_0 \) is stable. This raises the question under what conditions dynamical instability is possible in a quantum Hilbert space of finite dimensionality.

We thank Mark Dykman for useful discussions. This work is supported by the Israel science foundation and by the Israeli ministry of science.

[4] GA Prataviera and SS Mizrahi, “Many-particle...


