# Stability of the Grabert master equation 

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#### Abstract

We study the dynamics of a quantum system having Hilbert space of finite dimension $d_{\mathrm{H}}$. Instabilities are possible provided that the master equation governing the system's dynamics contain nonlinear terms. Here we consider the nonlinear master equation derived by Grabert. The dynamics near a fixed point is analyzed by using the method of linearization, and by evaluating the eigenvalues of the Jacobian matrix. We find that all these eigenvalues are non-negative, and conclude that the fixed point is stable. This finding raises the question: under what conditions instability is possible in a quantum system having finite $d_{\mathrm{H}}$ ?


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Consider a given closed quantum system having Hilbert space of finite dimension $d_{\mathrm{H}}$, whose master equation, which governs the time evolution of the reduced density matrix $\rho$, can be expressed as $\mathrm{d} \rho / \mathrm{d} t=\Theta(\rho)=$ $\Theta_{\mathrm{u}}(\rho)-\Theta_{\mathrm{d}}(\rho)$. The first term, which is given by $\Theta_{\mathrm{u}}(\rho)=(i / \hbar)[\rho, \mathcal{H}]$, where $\mathcal{H}=\mathcal{H}^{\dagger}$ is the Hamiltonian of the closed system, represents unitary evolution, and the second one $\Theta_{d}(\rho)$ represents the effect of coupling between the closed system and its environment. While it is commonly assumed that both the unitary term $\Theta_{u}(\rho)$ and the damping term $\Theta_{\mathrm{d}}(\rho)$ are linear in $\rho[1,2]$, in some cases the master equation can become nonlinear. Two types of nonlinearity are considered below [3, (4]. For the first one, which is henceforth referred to as unitary nonlinearity, the unitary term $\Theta_{\mathrm{u}}(\rho)$ is replaced by a nonlinear term. In most cases, unitary nonlinearity originates from either the mean field approximation [5-8], or from a transformation mapping the Hilbert space of finite dimension $d_{\mathrm{H}}$ into a space having infinite dimensionality (e.g. the Holstein-Primakoff transformation [9], which can yield a parametric instability in ferromagnetic resonators 10]). Here, we consider the second type, which is henceforth referred to as damping nonlinearity, and focus on the master equation that was proposed by Grabert [11], which has a damping term $\Theta_{\mathrm{d}}(\rho)$ nonlinear in $\rho$.

Grabert has shown that the invalidity of the quantum regression hypothesis gives rise to damping nonlinearity 11]. The nonlinear term added to the master equation ensures that the purity $\operatorname{Tr} \rho^{2}$ does not exceed unity [12, 13], and that entropy is generated at a non-negative rate, as is expected from the second law of thermodynamics (14]. Note, however, that under appropriate conditions, nonlinear dynamics may allow for faster than light signaling (15).

The Grabert master equation (GME) has a fixed point given by

$$
\begin{equation*}
\rho_{0}=\frac{e^{-\beta \mathcal{H}}}{\operatorname{Tr}\left(e^{-\beta \mathcal{H}}\right)}, \tag{1}
\end{equation*}
$$

where $\beta=1 /\left(k_{\mathrm{B}} T\right)$ is the inverse of the thermal energy [11]. At the fixed point $\rho_{0}$ the system is in thermal equilibrium having Boltzmann distribution.

Here we explore the stability of this fixed point $\rho_{0}$ for
the case where the Hamiltonian $\mathcal{H}$ of the closed system is time-independent. In a basis of energy eigenstates of a time-independent Hamiltonian both matrices $\mathcal{H}=$ $\operatorname{diag}\left(E_{1}, E_{2}, \cdots, E_{d_{\mathrm{H}}}\right)$ and $\rho_{0}=\operatorname{diag}\left(\rho_{1}, \rho_{2}, \cdots, \rho_{d_{\mathrm{H}}}\right)$ are diagonal, where $\rho_{n}=e^{-\beta E_{n}} / \operatorname{Tr}\left(e^{-\beta \mathcal{H}}\right)$ [see Eq. [ili].

For the case of thermal equilibrium, one may argue that the stability of $\rho_{0}$ is obvious. However, the stability of a driven system is anything but obvious. Note that in many cases the rotating wave approximation (RWA) is employed in order to model the dynamics of a given system under external driving, using a transformation into a rotating frame, in which the Hamiltonian becomes timeindependent in the RWA. Thus, our conclusion, that the fixed point $\rho_{0}$ is stable for any time independent Hermitian $\mathcal{H}$, can be extended beyond the limits of thermal equilibrium.

The GME for the reduced density matrix $\rho$ can be expressed as 12]

$$
\begin{equation*}
\frac{\mathrm{d} \rho}{\mathrm{~d} t}=\Theta(\rho)=\Theta_{\mathrm{u}}(\rho)-\Theta_{\mathrm{d}}(\rho), \tag{2}
\end{equation*}
$$

where the damping term is given by $\Theta_{\mathrm{d}}(\rho)=\Theta_{\mathrm{A}}(\rho)+$ $\Theta_{\mathrm{B}}(\rho)$, where $\Theta_{\mathrm{A}}(\rho)$, which is given by $\Theta_{\mathrm{A}}(\rho)=$ $\gamma_{\mathrm{E}}[Q,[Q, \rho]]$, is linear in $\rho$, and $\Theta_{\mathrm{B}}(\rho)$, which is given by $\Theta_{\mathrm{B}}(\rho)=\beta \gamma_{\mathrm{E}}\left[Q,[Q, \mathcal{H}]_{\rho}\right]$ is nonlinear. The constant $\gamma_{\mathrm{E}}>0$ is a damping rate, the Hermitian operator $Q^{\dagger}=Q$ describes the interaction between the quantum system and its environment, and

$$
\begin{equation*}
A_{\rho}=\int_{0}^{1} \mathrm{~d} \eta \rho^{\eta} A \rho^{1-\eta} \tag{3}
\end{equation*}
$$

Alternatively, the damping term $\Theta_{\mathrm{d}}(\rho)$ can be expressed as $\Theta_{\mathrm{d}}(\rho)=\beta \gamma_{\mathrm{E}}\left[Q,\left[Q, \mathcal{U}_{\mathrm{H}}\right]_{\rho}\right]$, where $\mathcal{U}_{\mathrm{H}}=$ $\mathcal{H}+\beta^{-1} \log \rho$ is the Helmholtz free energy operator [11]. According to the master equation (2), the time evolution of the Helmholtz free energy $\left\langle\mathcal{U}_{\mathrm{H}}\right\rangle=\operatorname{Tr}\left(\mathcal{U}_{\mathrm{H}} \rho\right)$ is governed by

$$
\begin{equation*}
\frac{\mathrm{d}\left\langle\mathcal{U}_{\mathrm{H}}\right\rangle}{\mathrm{d} t}=-\beta \gamma_{\mathrm{E}} \operatorname{Tr}\left(\mathcal{C}_{\rho} \mathcal{C}\right) \tag{4}
\end{equation*}
$$

where $\mathcal{C}=i\left[Q, \mathcal{U}_{\mathrm{H}}\right]$, and thus $\mathrm{d}\left\langle\mathcal{U}_{\mathrm{H}}\right\rangle / \mathrm{d} t \leq 0$ (since $\left.\mathcal{C}^{\dagger}=\mathcal{C}\right)$ 12, 16], i.e. the Helmholtz free energy $\left\langle\mathcal{U}_{\mathrm{H}}\right\rangle$ is a monotonically decreasing function of time.

Note that the operator $\mathcal{C}$ vanishes at the fixed point $\rho_{0}$ given by Eq. (11). Alternatively, the Kubo's identity [given by Eq. (4.2.17) of [16]] can be used to show that $\rho_{0}$ is a fixed point [11]. For some cases the existence of a limit cycle (i.e. periodic) solution for the GME (2) can be ruled out using Eq. (4). Along such a solution the condition $\mathcal{C}=0$ must be satisfied [since $\operatorname{Tr}\left(\mathcal{C}_{\rho} \mathcal{C}\right)=0$ implies that $\mathcal{C}=0$ when $\left.\operatorname{Tr} \rho^{2}<1\right]$. Hence, when $\rho=\rho_{0}$ is a unique solution of $\mathcal{C}=0$, a limit cycle solution can be ruled out.

A linear master equation can be derived by replacing the nonlinear term $\Theta_{\mathrm{B}}(\rho)$ by the term $\left(\beta^{\prime} / \hbar\right) \gamma_{\mathrm{E}}[Q,[Q, \mathcal{H}]]$, where $\beta^{\prime}>0$. It was shown in Ref. [17] (see also appendix B of Ref. [8]) that such a linear master equation is stable provided that $\gamma_{\mathrm{E}}>0$. Below we analyze the stability of the nonlinear GME (2).

The stability of the fixed point $\rho_{0}$ of the master equation (2) is explored by the method of linearization applied to the nonlinear term $\Theta_{\mathrm{B}}(\rho)$. In the vicinity of $\rho_{0}=\operatorname{diag}\left(\rho_{1}, \rho_{2}, \cdots, \rho_{d_{\mathrm{H}}}\right)$ the density matrix $\rho$ is expressed as $\rho=\rho_{0}+\epsilon \mathcal{V}$, where $\epsilon$ is a real small parameter. Let $u \rho u^{\dagger}=\rho_{\mathrm{d}}=\operatorname{diag}\left(\rho_{1}^{\prime}, \rho_{2}^{\prime}, \cdots, \rho_{d_{\mathrm{H}}}^{\prime}\right)$ be diagonal, where $u$ is unitary, i.e. $u^{\dagger} u=1$. With the help of time-independent perturbation theory one finds that the eigenvalues $\rho_{n}^{\prime}$ of $\rho$ are given by

$$
\begin{equation*}
\rho_{n}^{\prime}=\rho_{n}+\epsilon(n|\mathcal{V}| n)+O\left(\epsilon^{2}\right) \tag{5}
\end{equation*}
$$

and the unitary transformation $u$ that diagonalizes $\rho$ is given by

$$
\begin{equation*}
\left.\left.\left.u=\sum_{n}(\mid n)+\sum_{k \neq n} \frac{\epsilon(k|\mathcal{V}| n)}{\rho_{n}-\rho_{k}} \right\rvert\, k\right)\right)\left(n \mid+O\left(\epsilon^{2}\right)\right. \tag{6}
\end{equation*}
$$

or

$$
\begin{equation*}
u=1-i \epsilon F+O\left(\epsilon^{2}\right) \tag{7}
\end{equation*}
$$

where the Hermitian matrix $F$ is given by

$$
\begin{equation*}
\left.\left.F=\sum_{k \neq l} \frac{i(k|\mathcal{V}| l)}{\rho_{l}-\rho_{k}} \right\rvert\, k\right)(l \mid, \tag{8}
\end{equation*}
$$

$(k|\mathcal{V}| l)=\mathcal{V}_{k l}$ is the ( $k$ 'th raw - l'th column) matrix element of $\mathcal{V}$, and $\mid k)\left(l \mid\right.$ denotes a $d_{\mathrm{H}} \times d_{\mathrm{H}}$ matrix having entry 1 in the ( $k$ 'th raw - $l$ 'th column), and entry 0 elsewhere.

Using the identity [12]

$$
\begin{equation*}
\int_{0}^{1} x^{\eta} y^{1-\eta} \mathrm{d} \eta=\mathcal{F}(x, y) \tag{9}
\end{equation*}
$$

where

$$
\begin{equation*}
\mathcal{F}(x, y)=\frac{x-y}{\log x-\log y} \tag{10}
\end{equation*}
$$

one finds that (recall that $\rho_{\mathrm{d}}$ is diagonal)

$$
\begin{equation*}
\int_{0}^{1} \mathrm{~d} \eta \rho_{\mathrm{d}}^{\eta} A \rho_{\mathrm{d}}^{1-\eta}=\mathcal{F}^{\prime} \circ A \tag{11}
\end{equation*}
$$

where o denotes the Hadamard matrix multiplication (element by element matrix multiplication), and where the matrix elements of $\mathcal{F}^{\prime}$ are given by $\mathcal{F}_{n m}^{\prime}=\mathcal{F}\left(\rho_{n}^{\prime}, \rho_{m}^{\prime}\right)$. Note that $\mathcal{F}_{n m}^{\prime}=\mathcal{F}_{n m}+O(\epsilon)$, where $\mathcal{F}_{n m}=\mathcal{F}\left(\rho_{n}, \rho_{m}\right)$ [see Eq. (5)], hence, the following holds [see Eqs. (3) and (77) and note that $\left.u A u^{\dagger}=A+i \epsilon[A, F]+O\left(\epsilon^{2}\right)\right]$

$$
\begin{equation*}
A_{\rho}=\mathcal{F}^{\prime} \circ A+i \epsilon[F, \mathcal{F} \circ A]+i \epsilon \mathcal{F} \circ[A, F]+O\left(\epsilon^{2}\right) \tag{12}
\end{equation*}
$$

where $\mathcal{F}^{\prime}=\mathcal{F}+\epsilon(\mathrm{d} \mathcal{F} / \mathrm{d} \epsilon)+O\left(\epsilon^{2}\right)$.
The following holds [see Eq. (10)]

$$
\begin{equation*}
\mathcal{F}(x, y)=\frac{x+y}{2} f_{\mathrm{D}}\left(\frac{x-y}{x+y}\right) \tag{13}
\end{equation*}
$$

where the function $f_{\mathrm{D}}(\eta)$ is given by

$$
\begin{equation*}
f_{\mathrm{D}}(\eta)=\frac{2 \eta}{\log \frac{1+\eta}{1-\eta}}=\frac{\eta}{\tanh ^{-1} \eta} \tag{14}
\end{equation*}
$$

The function $f_{\mathrm{D}}$ is symmetric, i.e. $f_{\mathrm{D}}(-\eta)=f_{\mathrm{D}}(\eta)$, and the following holds $f_{\mathrm{D}}(0)=1$ and $f_{\mathrm{D}}( \pm 1)=0$. With the help of Eqs. (5) and (13) one finds that the matrix $\mathrm{d} \mathcal{F} / \mathrm{d} \epsilon$ is real, symmetric, and the following holds (no summation due to repeated indices $n$ and $m$ )

$$
\begin{equation*}
\left(\frac{\mathrm{d} \mathcal{F}}{\mathrm{~d} \epsilon}\right)_{n m}=\frac{\mathrm{d} \alpha_{n m}}{\mathrm{~d} \epsilon} F_{n m}+\alpha_{n m} \frac{\mathrm{~d} \eta_{n m}}{\mathrm{~d} \epsilon} F_{n m}^{\prime} \tag{15}
\end{equation*}
$$

where $\alpha_{n m}=\left(\rho_{n}+\rho_{m}\right) / 2, \quad \eta_{n m}=$ $\left(\rho_{n}-\rho_{m}\right) /\left(\rho_{n}+\rho_{m}\right), F_{n m}=f_{\mathrm{D}}\left(\eta_{n m}\right)$, and where $F_{n m}^{\prime}=f_{\mathrm{D}}^{\prime}\left(\eta_{n m}\right)$. Moreover, $\operatorname{Tr}(\mathrm{d} \mathcal{F} / \mathrm{d} \epsilon)=0$ (note that $F_{n n}=1$ and $F_{n n}^{\prime}=0$ ).

The $d_{\mathrm{H}}^{2}-1$ Hermitian and trace-less $d_{\mathrm{H}} \times d_{\mathrm{H}}$ generalized Gell-Mann matrices $\lambda_{n}$, which span the $\operatorname{SU}\left(d_{\mathrm{H}}\right)$ Lie algebra, satisfy the orthogonality relation

$$
\begin{equation*}
\frac{\operatorname{Tr}\left(\lambda_{a} \lambda_{b}\right)}{2}=\delta_{a b} \tag{16}
\end{equation*}
$$

For the case $d_{\mathrm{H}}=2\left(d_{\mathrm{H}}=3\right)$ the matrices are called Pauli (Gell-Mann) matrices. The set $\left\{\lambda_{a}\right\}$ of $d_{\mathrm{H}}^{2}-1$ matrices can be divided into three subsets. The subset $\left\{\lambda_{\mathrm{X},(n, m)}\right\}$ contains $d_{\mathrm{H}}\left(d_{\mathrm{H}}-1\right) / 2$ matrices given by $\left.\lambda_{\mathrm{X},(n, m)}=\mid n\right)(m|+| m)\left(n \mid\right.$, and the subset $\left\{\lambda_{\mathrm{Y},(n, m)}\right\}$ contains $d_{\mathrm{H}}\left(d_{\mathrm{H}}-1\right) / 2$ matrices given by $\lambda_{\mathrm{Y},(n, m)}=$ $-i \mid n)(m|+i| m)\left(n \mid\right.$, where $1 \leq m<n \leq d_{\mathrm{H}}$. The subset $\left\{\lambda_{\mathrm{Z}, l}\right\}$ contains $d_{\mathrm{H}}-1$ diagonal matrices given by

$$
\begin{equation*}
\lambda_{\mathrm{z}, l}=\sqrt{\frac{2}{l(l+1)}}(-l \mid l+1)\left(l+1\left|+\sum_{j=1}^{l}\right| j\right)(j \mid) \tag{17}
\end{equation*}
$$

where $1 \leq l \leq d_{\mathrm{H}}-1$.

It is convenient to express the perturbation $\epsilon \mathcal{V}=\rho-$ $\rho_{0}$ as $\epsilon \mathcal{V}=\bar{\kappa} \cdot \bar{\lambda}$, where $\bar{\kappa}=\left(\kappa_{1}, \kappa_{2}, \cdots, \kappa_{d_{\mathrm{H}}^{2}-1}\right)$ and $\bar{\lambda}=\left(\lambda_{1}, \lambda_{2}, \cdots, \lambda_{d_{\mathrm{H}}^{2}-1}\right)$. In this notation the GME (2) becomes (repeated index implies summation)

$$
\begin{equation*}
\frac{\mathrm{d} \kappa_{b}}{\mathrm{~d} t} \lambda_{b}=\Theta\left(\rho_{0}+\kappa_{b} \lambda_{b}\right) \tag{18}
\end{equation*}
$$

or [see Eq. (16)]

$$
\begin{equation*}
\frac{\mathrm{d} \kappa_{a}}{\mathrm{~d} t}=\frac{1}{2} \operatorname{Tr}\left(\Theta\left(\rho_{0}+\kappa_{b} \lambda_{b}\right) \lambda_{a}\right) \tag{19}
\end{equation*}
$$

To first order in $\bar{\kappa}$

$$
\begin{equation*}
\frac{\mathrm{d} \kappa_{a}}{\mathrm{~d} t}=\frac{1}{2} \operatorname{Tr}\left(\frac{\partial \Theta}{\partial \kappa_{b}} \lambda_{a} \kappa_{b}\right) \tag{20}
\end{equation*}
$$

or in a vector form

$$
\begin{equation*}
\frac{\mathrm{d} \bar{\kappa}}{\mathrm{~d} t}=J \bar{\kappa} \tag{21}
\end{equation*}
$$

where the Jacobian matrix $J$ is given by $J=J_{\mathrm{u}}-J_{\mathrm{A}}-J_{\mathrm{B}}$, and where

$$
\begin{equation*}
J_{\Sigma}=\frac{1}{2} \operatorname{Tr}\left(\frac{\partial \Theta_{\Sigma}}{\partial \kappa_{b}} \lambda_{a}\right) \tag{22}
\end{equation*}
$$

with $\Sigma \in\{\mathrm{u}, \mathrm{A}, \mathrm{B}\}$.
The system's stability depends on the set of eigenvalues of the Jacobian matrix $J$, which is denoted by $\mathcal{S}$. The system is stable provided that real $(\xi)<0$ for any $\xi \in \mathcal{S}$. It was shown in appendix B of Ref. [8] that such a system is stable provided that $J_{\mathrm{u}}, J_{\mathrm{A}}$ and $J_{\mathrm{B}}$ are all real, $J_{\mathrm{u}}$ is antisymmetric, all diagonal elements of $J_{\mathrm{A}}+J_{\mathrm{B}}$ are positive, and $d_{\mathrm{H}}$ is finite. Properties of the matrices $J_{\mathrm{u}}$, $J_{\mathrm{A}}$ and $J_{\mathrm{B}}$ are analyzed below.

The matrix $J_{\mathrm{u}}$, which governs the unitary evolution, is given by [recall the trace identity $\operatorname{Tr}(X Y)=\operatorname{Tr}(Y X)]$

$$
\begin{align*}
J_{\mathrm{u}} & =\frac{i}{2 \hbar} \operatorname{Tr}\left(\left[\lambda_{b}, \mathcal{H}\right] \lambda_{a}\right) \\
& =\frac{i}{2 \hbar} \operatorname{Tr}\left(\mathcal{H}\left[\lambda_{a}, \lambda_{b}\right]\right) \tag{23}
\end{align*}
$$

hence $J_{\mathrm{u}}$ is real and antisymmetric provided that $\mathcal{H}$ is Hermitian (note that $i\left[\lambda_{b}, \lambda_{a}\right]$ is Hermitian).

The matrix $J_{\mathrm{A}}$ is given by

$$
\begin{align*}
J_{\mathrm{A}} & =\frac{\gamma_{\mathrm{E}}}{2} \operatorname{Tr}\left(\left[Q,\left[Q, \lambda_{b}\right]\right] \lambda_{a}\right) \\
& =\frac{\gamma_{\mathrm{E}}}{2} \operatorname{Tr}\left(-\left[Q, \lambda_{b}\right]\left[Q, \lambda_{a}\right]\right) \tag{24}
\end{align*}
$$

Both matrices $i\left[Q, \lambda_{a}\right]$ and $i\left[Q, \lambda_{b}\right]$ are Hermitian, provided that $Q$ is Hermitian, hence $J_{\mathrm{A}}$ is real (recall that $\gamma_{\mathrm{E}}$ is positive). The diagonal elements of $J_{\mathrm{A}}$ are positive since $-\left[Q, \lambda_{b}\right]\left[Q, \lambda_{a}\right]$ is positive-definite for the case $a=b$.

The diagonal elements of the matrix $J_{\mathrm{B}}$ cab be evaluated using the linearization of the term $A_{\rho}$ given by Eq. (12). For the case where the perturbation $\mathcal{V}=\left(\rho-\rho_{0}\right) / \epsilon$ is a generalized Gell-Mann matrix, i.e. $\mathcal{V} \in\left\{\lambda_{a}\right\}$, the following holds [see Eq. (8)]

$$
F=\left\{\begin{array}{cl}
\frac{\lambda_{\mathrm{Y},(n, m)}}{\rho_{n}-\rho_{m}} & \text { if } \mathcal{V}=\lambda_{\mathrm{X},(n, m)}  \tag{25}\\
-\frac{\lambda_{\mathrm{X},(n, m)}}{\rho_{n}-\rho_{m}} & \text { if } \mathcal{V}=\lambda_{\mathrm{Y},(n, m)}
\end{array},\right.
$$

and [see Eq. (12), and note that, according to Eq. (5), $\mathcal{F}^{\prime}=\mathcal{F}+O\left(\epsilon^{2}\right)$ when all diagonal elements of the perturbation vanish, e.g. when $\mathcal{V} \in\left\{\lambda_{\mathrm{X},(n, m)}\right\} \cup\left\{\lambda_{\mathrm{Y},(n, m)}\right\}$, and, according to Eqs. (7) and (18), $u=1+O\left(\epsilon^{2}\right)$ when the perturbation is diagonal, e.g. when $\left.\mathcal{V} \in\left\{\lambda_{\mathrm{Z}, l}\right\}\right]$
$\frac{\mathrm{d} A_{\rho}}{\mathrm{d} \epsilon}=\left\{\begin{array}{cl}\frac{\left[\mathcal{F} \circ A, \lambda_{\mathrm{Y},(n, m)}\right]-\mathcal{F} \circ\left[A, \lambda_{\mathrm{Y},(n, m)}\right]}{i\left(\rho_{n}-\rho_{m}\right)} & \text { if } \mathcal{V}=\lambda_{\mathrm{X},(n, m)} \\ \frac{\left[\mathcal{F} \circ A, \lambda_{\mathrm{X},(n, m)}\right]-\mathcal{F} \circ\left[A, \lambda_{\mathrm{X},(n, m)}\right]}{(-i)\left(\rho_{\rho}-\rho_{m}\right)} & \text { if } \mathcal{V}=\lambda_{\mathrm{Y},(n, m)} \\ \frac{\mathrm{d} \mathcal{F}^{\prime}}{\mathrm{d} \epsilon} \circ A & \text { if } \mathcal{V}=\lambda_{\mathrm{Z},(n, m)}\end{array}\right.$.
The diagonal elements of $J_{\mathrm{A}}+J_{\mathrm{B}}$ are evaluated by using of Eq. (26) with different values of the perturbation $\mathcal{V}$.

The diagonal matrix element corresponding to the generalized Gell-Mann matrix $\lambda_{\mathrm{Z}, l}$, which is labeled by $j_{l}$, is given by [see Eqs. (22), (24) and (26)]

$$
\begin{align*}
j_{l} & =\frac{\gamma_{\mathrm{E}}}{2} \operatorname{Tr}\left(-\left[Q, \lambda_{\mathrm{z}, l}\right]^{2}\right) \\
& +\frac{\beta \gamma_{\mathrm{E}}}{2} \operatorname{Tr}\left(\left[Q, \frac{\mathrm{~d} \mathcal{F}}{\mathrm{~d} \epsilon} \circ[Q, \mathcal{H}]\right] \lambda_{\mathrm{Z}, l}\right) \tag{27}
\end{align*}
$$

where the term $\mathrm{d} \mathcal{F} / \mathrm{d} \epsilon$ is evaluated according to Eq. (15) for the case where the perturbation is given by $\mathcal{V}=\lambda_{\mathrm{Z}, l}$. In terms of the elements of the diagonal matrix $\lambda_{\mathrm{Z}, l}=$ $\operatorname{diag}\left(\nu_{1}, \nu_{2}, \cdots, \nu_{d_{\mathrm{H}}}\right)$ one finds using Eq. (5) that $\rho_{n}^{\prime}=$ $\rho_{n}+\epsilon \nu_{n}+O\left(\epsilon^{2}\right)$, hence $(\mathrm{d} \mathcal{F} / \mathrm{d} \epsilon)_{n m}=d_{n m}$, where

$$
\begin{equation*}
d_{n m}=\frac{\nu_{n m} F_{n m}}{2 \varkappa_{n m}}\left(1+\frac{\left(\varkappa_{n m}-\eta_{n m}\right) F_{n m}^{\prime}}{F_{n m}}\right) \tag{28}
\end{equation*}
$$

$\nu_{n m}=\nu_{n}-\nu_{m}$ and $\varkappa_{n m}=\left(\nu_{n}-\nu_{m}\right) /\left(\nu_{n}+\nu_{m}\right)$. The following holds $d_{n m}=d_{m n}$, hence Eq. (27) yields

$$
\begin{equation*}
j_{l}=\gamma_{\mathrm{E}} \sum_{n<m} \zeta_{n m} \nu_{n m}^{2}\left|q_{n m}\right|^{2} \tag{29}
\end{equation*}
$$

where $\zeta_{n m}=1+d_{n m} e_{n m} / \nu_{n m}, e_{n m}=\beta\left(E_{n}-E_{m}\right)$, and where $q_{n m}$ are the matrix elements of the operator $Q$ (recall that it is assumed that $Q^{\dagger}=Q$, i.e. $q_{m n}=q_{n m}^{*}$ ). With the help of the relation $\eta_{n m}=-\tanh \left(e_{n m} / 2\right)$ [see Eq. (11)] one finds that $\zeta_{n m}=\zeta\left(\eta_{n m}, \varkappa_{n m}\right)$, where the function $\zeta(\eta, \varkappa)$ is given by [see Eq. (14) and note that $\left.1-\left(1 /\left(1-\eta^{2}\right)\right)\left(\eta / \tanh ^{-1} \eta\right)=\eta F^{\prime}(\eta) / F(\eta)\right]$

$$
\begin{equation*}
\zeta(\eta, \varkappa)=\frac{f_{\mathrm{D}}(\eta)}{1-\eta^{2}}\left(1-\frac{\eta}{\varkappa}\right) \tag{30}
\end{equation*}
$$

The following holds [see Eq. (17), and note that only the cases for which $v_{n m} \neq 0$, i.e. the cases that can contribute to $j_{l}$, are listed]

$$
-\frac{1}{\varkappa_{n m}}=\left\{\begin{array}{cc}
\frac{l-1}{l+1} & n \leq l \text { and } m=l+1  \tag{31}\\
1 & n \leq l \text { and } m>l+1 \\
1 & n=l+1 \text { and } m>l+1
\end{array}\right.
$$

hence $0 \leq(-1 / \varkappa) \leq 1$ for all terms contributing to $j_{l}$, hence $\zeta_{n m} \nu_{n m}^{2} \geq 0$ for these terms, and consequently $j_{l} \geq 0$.

The diagonal matrix element corresponding to the generalized Gell-Mann matrix $\lambda_{\mathrm{X},(2,1)}\left(\lambda_{\mathrm{Y},(2,1)}\right)$ is labelled by $j_{\mathrm{X}}\left(j_{\mathrm{Y}}\right)$. We show below that both $j_{\mathrm{X}}$ and $j_{\mathrm{Y}}$ are nonnegative. The proof is applicable for all other diagonal elements, corresponding to all generalized Gell-Mann matrices $\lambda \in\left\{\lambda_{\mathrm{X},(n, m)}\right\} \cup\left\{\lambda_{\mathrm{Y},(n, m)}\right\}$ with $(n, m) \neq(2,1)$, since the ordering of the energy eigenvectors is arbitrary.

With the help of Eqs. (22), (24) and (26) one finds that [the subscript $(2,1)$ is omitted for brevity]

$$
\begin{align*}
\frac{j_{\mathrm{X}}}{\frac{\gamma_{\mathrm{E}}}{2}} & =\operatorname{Tr}\left(-\left[Q, \lambda_{\mathrm{X}}\right]\left[Q, \lambda_{\mathrm{X}}\right]\right) \\
& +\operatorname{Tr}\left(\beta\left[Q, \frac{\left[\mathcal{F} \circ[Q, \mathcal{H}], \lambda_{\mathrm{Y}}\right]-\mathcal{F} \circ\left[[Q, \mathcal{H}], \lambda_{\mathrm{Y}}\right]}{i\left(\rho_{2}-\rho_{1}\right)}\right] \lambda_{\mathrm{X}}\right) \tag{32}
\end{align*}
$$

and

$$
\begin{align*}
\frac{j_{\mathrm{Y}}}{\frac{\gamma_{\mathrm{E}}}{2}} & =\operatorname{Tr}\left(-\left[Q, \lambda_{\mathrm{Y}}\right]\left[Q, \lambda_{\mathrm{Y}}\right]\right) \\
& +\operatorname{Tr}\left(\beta\left[Q, \frac{\left[\mathcal{F} \circ[Q, \mathcal{H}], \lambda_{\mathrm{X}}\right]-\mathcal{F} \circ\left[[Q, \mathcal{H}], \lambda_{\mathrm{X}}\right]}{(-i)\left(\rho_{2}-\rho_{1}\right)}\right] \lambda_{\mathrm{Y}}\right) \tag{33}
\end{align*}
$$

hence

$$
\begin{equation*}
\frac{j_{\mathrm{X}}}{\gamma_{\mathrm{E}}}=q_{\mathrm{d}}^{2}+4 v q_{12}^{\prime \prime 2}+\sum_{n=1}^{2} \sum_{m \geq 3} G_{n m}\left|q_{n m}\right|^{2} \tag{34}
\end{equation*}
$$

and

$$
\begin{equation*}
\frac{j_{\mathrm{Y}}}{\gamma_{\mathrm{E}}}=q_{\mathrm{d}}^{2}+4 v q_{12}^{\prime 2}+\sum_{n=1}^{2} \sum_{m \geq 3} G_{n m}\left|q_{n m}\right|^{2} \tag{35}
\end{equation*}
$$

where $q_{\mathrm{d}}=q_{11}-q_{22}$,

$$
\begin{equation*}
v=1-\frac{\left(\mathcal{F}_{11}+\mathcal{F}_{22}-2 \mathcal{F}_{12}\right) e_{12}}{2\left(\rho_{1}-\rho_{2}\right)} \tag{36}
\end{equation*}
$$

$q_{12}^{\prime}=\operatorname{Re} q_{12}, q_{12}^{\prime \prime}=\operatorname{Im} q_{12}$, and where

$$
\begin{equation*}
G_{n m}=1+\frac{\left(\mathcal{F}_{1 m}-\mathcal{F}_{2 m}\right) e_{n m}}{\rho_{1}-\rho_{2}} \tag{37}
\end{equation*}
$$

With the help of Eqs. (11), (13) and (14) one finds that $\left[\right.$ note that $e_{n m}=-\log \left(\rho_{n} / \rho_{m}\right)=$ $\left.\log \left(\left(1-\eta_{n m}\right) /\left(1+\eta_{n m}\right)\right)=-2 \eta_{n m} / f_{\mathrm{D}}\left(\eta_{n m}\right)\right]$

$$
\begin{equation*}
v=\frac{1}{f_{\mathrm{D}}\left(\eta_{12}\right)} \tag{38}
\end{equation*}
$$

and that $G_{1 m}=G\left(\rho_{1} / \rho_{m}, \rho_{2} / \rho_{m}\right)$ and $G_{2 m}=$ $G\left(\rho_{2} / \rho_{m}, \rho_{1} / \rho_{m}\right)$, where the function $G$ is given by

$$
\begin{equation*}
G\left(r_{1}, r_{2}\right)=1-\frac{\frac{r_{1}-1}{\log r_{1}}-\frac{r_{2}-1}{\log r_{2}}}{r_{1}-r_{2}} \log r_{1} \tag{39}
\end{equation*}
$$

or

$$
\begin{equation*}
G\left(r_{1}, r_{2}\right)=\frac{r_{2}-1}{r_{2} \log r_{2}} \frac{\log \frac{r_{1}}{r_{2}}}{\frac{r_{1}}{r_{2}}-1} \tag{40}
\end{equation*}
$$

hence $v \geq 1\left[\right.$ since $\left.0 \leq f_{\mathrm{D}}\left(\eta_{12}\right) \leq 1\right]$ and $G_{n m} \geq 0$ [see Eq. (40), and note that for non-negative $r_{1}$ and $r_{2}$, both the first factor, which depends on $r_{2}$ only, and the second one, which depends on $r_{1} / r_{2}$ only, are non-negative], and thus both $j_{\mathrm{X}}$ and $j_{\mathrm{Y}}$ are non-negative.

In summary, the dynamics governed by the GME (2) in the vicinity of the steady state $\rho_{0}$ depends on the $d_{\mathrm{H}}^{2}-1$ diagonal element of the Jacobean matrix $J_{\mathrm{A}}+J_{\mathrm{B}}$. Our derived expressions for the eigenvalues, given by Eqs. (29), (34) and (35), can be used to evaluate statistical properties of the system near its steady state $\rho_{0}$. We find that all these eigenvalues are non-negative, and conclude that the steady state $\rho_{0}$ is stable. This raises the question under what conditions dynamical instability is possible in a quantum Hilbert space of finite dimensionality.

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[1] Bernd Fernengel and Barbara Drossel, "Bifurcations and chaos in nonlinear lindblad equations", Journal of Physics A: Mathematical and Theoretical, vol. 53, no. 38, pp. 385701, 2020.
[2] Goran Lindblad, "On the generators of quantum dy-
namical semigroups", Communications in Mathematical Physics, vol. 48, no. 2, pp. 119-130, 1976.
[3] VI Yukalov, "Nonlinear spin dynamics in nuclear magnets", Physical Review B, vol. 53, no. 14, pp. 9232, 1996.
[4] GA Prataviera and SS Mizrahi, "Many-particle
sudarshan-lindblad equation: mean-field approximation, nonlinearity and dissipation in a spin system", Revista Brasileira de Ensino de Física, vol. 36, no. 4, pp. 01-11, 2014.
[5] Heinz-Peter Breuer, Francesco Petruccione, et al., The theory of open quantum systems, Oxford University Press on Demand, 2002.
[6] Barbara Drossel, "What condensed matter physics and statistical physics teach us about the limits of unitary time evolution", Quantum Studies: Mathematics and Foundations, vol. 7, no. 2, pp. 217-231, 2020.
[7] C Hicke and MI Dykman, "Classical dynamics of resonantly modulated large-spin systems", Physical Review $B$, vol. 78, no. 2, pp. 024401, 2008.
[8] Roei Levi, Sergei Masis, and Eyal Buks, "Instability in the hartmann-hahn double resonance", Phys. Rev. A, vol. 102, pp. 053516, Nov 2020.
[9] T Holstein and Hl Primakoff, "Field dependence of the intrinsic domain magnetization of a ferromagnet", Physical Review, vol. 58, no. 12, pp. 1098, 1940.
[10] E Schlömann, JJ Green, and uU Milano, "Recent developments in ferromagnetic resonance at high power levels", Journal of Applied Physics, vol. 31, no. 5, pp. S386S395, 1960.
[11] H Grabert, "Nonlinear relaxation and fluctuations of damped quantum systems", Zeitschrift für Physik B Condensed Matter, vol. 49, no. 2, pp. 161-172, 1982.
[12] Hans Christian Öttinger, "Nonlinear thermodynamic quantum master equation: Properties and examples", Physical Review A, vol. 82, no. 5, pp. 052119, 2010.
[13] Hans Christian Öttinger, "The geometry and thermodynamics of dissipative quantum systems", EPL (Europhysics Letters), vol. 94, no. 1, pp. 10006, 2011.
[14] David Taj and Hans Christian Öttinger, "Natural approach to quantum dissipation", Physical Review A, vol. 92, no. 6, pp. 062128, 2015.
[15] Angelo Bassi and Kasra Hejazi, "No-faster-than-lightsignaling implies linear evolution. a re-derivation", European Journal of Physics, vol. 36, no. 5, pp. 055027, 2015.
[16] Ryogo Kubo, Morikazu Toda, and Natsuki Hashitsume, Statistical physics II: nonequilibrium statistical mechanics, vol. 31, Springer Science \& Business Media, 2012.
[17] Herbert Spohn, "An algebraic condition for the approach to equilibrium of an open n-level system", Letters in Mathematical Physics, vol. 2, no. 1, pp. 33-38, 1977.

