Disentanglement by deranking and by suppression of correlation

Eyal Buks*
Andrew and Erna Viterbi Department of Electrical Engineering, Technion, Haifa 32000, Israel
(Dated: February 8, 2026)

The spontaneous disentanglement hypothesis is motivated by some outstanding issues in standard quantum mechanics, including the problem of quantum measurement. The current study compares between some possible methods that can be used to implement the hypothesis. Disentanglement is formulated using a nonlinear operator, which can be used to modify both the Schrödinger equation for the quantum state vector, and the master equation for the density operator. Two types of nonlinear disentanglement operators are explored. The first one gives rise to matrix deranking, and the second one to correlation suppression. Both types are demonstrated using a two spin system that is driven close to the Hartmann–Hahn double resonance. It is shown that limit cycle steady state solutions, which are excluded by standard quantum mechanics, become possible in the presence of disentanglement.

I. INTRODUCTION

Time evolution in standard quantum mechanics (QM) is governed by linear equations of motion. Some outstanding issues in QM have motivated the study of a variety of nonlinear extensions to the standard formulation of QM [1–8]. To address the problem of quantum measurement [9–11], nonlinear extensions that give rise to spontaneous collapse of the state vector have been proposed [12–16]. Moreover, multistability in finite systems, which is excluded by standard QM [17–22], can be theoretically accounted for, provided that nonlinearity is permitted.

The current study explores some possible nonlinear extensions that give rise to disentanglement [23]. The hypothesis that disentanglement spontaneously occurs in quantum systems is motivated by both the problem of quantum measurement and by the difficultly to account for multi–stabilities in standard QM [24]. Several different possibilities to implement the hypothesis are introduced and compared below.

Consider a modified Schrödinger equation having a form given by [25, 26]

$$\frac{\mathrm{d}}{\mathrm{d}t}\left|\psi\right\rangle = \left[-i\hbar^{-1}\mathcal{H} - \left(\Theta - \frac{\left\langle\psi\right|\Theta\left|\psi\right\rangle}{\left\langle\psi\right|\psi\rangle}\right)\right]\left|\psi\right\rangle \ , \tag{1}$$

where $\hbar$ is the Planck's constant, and $\mathcal{H}$ is the Hamiltonian. The nonlinear extension in Eq. (1) depends on the Hermitian operator Θ, which is allowed to be dependent on the state vector $\left|\psi\right\rangle$. The corresponding master equation for the pure state density operator $\rho = \left|\psi\right\rangle\left\langle\psi\right|$ is given by [5, 7, 27–29]

$$\frac{\mathrm{d}\rho}{\mathrm{d}t} = i\hbar^{-1}\left[\rho,\mathcal{H}\right] - \Theta\rho - \rho\Theta + 2\left\langle\Theta\right\rangle\frac{\rho}{\mathrm{Tr}\,\rho} \ , \tag{2}$$

where $\left\langle\Theta\right\rangle = \mathrm{Tr}\left(\Theta\rho\right)$. Norm is conserved by both the modified Schrödinger equation (1) and the modified master equation (2) [note that Eq. (1) yields $(\mathrm{d}/\mathrm{d}t)\left\langle\psi\right|\psi\rangle = 0$, and Eq. (2) yields $(\mathrm{d}/\mathrm{d}t)\,\mathrm{Tr}\,\rho = 0$]. Moreover, positivity of the density matrix ρ is conserved by the modified master equation (2) [see Eq. (2.201) of Ref. [30]]. In the current study, both modified Schrödinger equation (1) and modified master equation (2) are employed to explore the process of disentanglement [31].

The time evolution generated by the nonlinear master equation (2) can be expressed as (it is assumed that $\mathrm{Tr}\,\rho = 1$)

$$\rho\left(t+\tau\right) = \sum_{k\in\{0,1\}} K_k\rho\left(t\right)K_k^{\dagger} + O\left(\tau^2\right) \ , \tag{3}$$

where the Kraus operators K_0 and K_1, which are given by $K_0 = 1 - \left(i\hbar^{-1}\mathcal{H} + \Theta\right)\tau$ and $K_1 = \sqrt{2\left\langle\Theta\right\rangle\tau}$, satisfy the norm conservation condition $\left\langle K_0^{\dagger}K_0 + K_1^{\dagger}K_1\right\rangle = 1 + O\left(\tau^2\right)$. Note that, similarly to the process of dephasing of standard QM [e.g. see Eq. (17) of Ref. [32]], for the case $\mathcal{H} = 0$, the Kraus operators K_0 and K_1 are both Hermitian. However, while standard dephasing occurs in a fixed basis, which is determined by the coupling between a quantum system and its environment, the basis associated with the nonlinear master equation (1), which is made of eigenvectors of the state–dependent operator Θ, is not fixed.

The nonlinear extension [in both the modified Schrödinger equation (1) and the modified master equation (2)] can be employed to suppress any given physical property, provided that $\left\langle\Theta\right\rangle$ quantifies that property. For example, thermalization can be introduced by taking Θ to be given by $\Theta = \gamma_{\mathrm{H}}\beta\mathcal{U}_{\mathrm{H}}$, where $\mathcal{U}_{\mathrm{H}} = \mathcal{H} + \beta^{-1}\log\rho$ is the Helmholtz free energy operator, the real parameter γ_{H} represents the rate of thermalization, $\beta = 1/\left(k_{\mathrm{B}}T\right)$ is the thermal energy inverse, k_{B} is the Boltzmann's constant, and T is the temperature [33, 34]. The thermalization process can be described in terms of the normalized rank of the density matrix ρ. For a general $D\times D$ positive semi–definite (PSD) matrix A, the normalized rank $\mathcal{R}\left(A\right)$ of A is defined by [35]

$$\mathcal{R}\left(A\right) = \frac{1}{\log D}\mathrm{Tr}\left(-\frac{A}{\mathrm{Tr}\,A}\log\frac{A}{\mathrm{Tr}\,A}\right) \ . \tag{4}$$

*Electronic address: eyal@ee.technion.ac.il

Note that the normalized rank $\mathcal{R}(A)$ is generally bounded by $\mathcal{R}(A) \in [0,1]$. The entropy expectation value $\langle -\log\rho\rangle = \mathrm{Tr}(-\rho\log\rho)$ is related to the normalized rank $\mathcal{R}(\rho)$ of the density matrix ρ by $\langle -\log\rho\rangle = \log(D)\,\mathcal{R}(\rho)$, where D is the Hilbert space dimensionality (which is assumed to be finite). As is discussed below, in a similar way, disentanglement can be described as a matrix deranking process.

Disentanglement can be introduced provided that $\langle\Theta\rangle$ quantifies the level of entanglement associated with the state vector $|\psi\rangle$ and/or the density operator ρ [27, 36–49]. Two different types of nonlinear disentanglement operators Θ are explored in the current study. Operators Θ belonging to the first type give rise to matrix deranking, whereas correlation between subsystems is suppressed by operators Θ belonging to the second type [31] (see appendix F). Two alternative ways, which are based on deranking (i.e. suppression of matrix normalized rank) are proposed and explored below. Deranking of the state matrix is discussed in appendix B, whereas appendix C is devoted to deranking of the Bloch matrix [50]. For a pure state, both deranking methods are applicable, whereas only Bloch matrix deranking is applicable for general (i.e. mixed) states. Damping is accounted for using both the Gorini-Kossakowski-Sudarshan-Lindblad (GKSL) master equation [51] (see appendix A), and the Schrödinger–Langevin equation [52–54] (see appendix E). All proposed methods to introduce disentanglement are explored using a two spin system, which is driven close to the Hartmann–Hahn double resonance [55], and which is described in the next section.

II. TWO-SPIN SYSTEM

Consider a system composed of two spin 1/2 particles [see Fig. 1(a)]. The first spin, which is labelled as 'a', has a relatively low Larmor angular frequency ω_{a} in comparison with the Larmor angular frequency ω_{b} of the second spin, which is labelled as 'b', and which is externally driven. The angular momentum vector operator of spin a (b) is denoted by $\mathbf{S}_{\mathrm{a}}$ ($\mathbf{S}_{\mathrm{b}}$). The Hamiltonian $\mathcal{H}$ of the closed two-spin system is given by

$$\mathcal{H} = \omega_{\mathrm{a}} S_{\mathrm{az}} + \omega_{\mathrm{b}} S_{\mathrm{bz}} + \frac{\omega_1 (S_{\mathrm{b}+} + S_{\mathrm{b}-})}{2} + V\ , \tag{5}$$

where the driving amplitude and angular frequency are denoted by ω_1 and $\omega_{\mathrm{p}} = \omega_{\mathrm{b}} + \Delta$, respectively ($\Delta$ is the driving detuning angular frequency), the operators $S_{\mathrm{a}\pm}$ are given by $S_{\mathrm{a}\pm} = S_{\mathrm{ax}} \pm iS_{\mathrm{ay}}$, and the rotated operators $S_{\mathrm{b}\pm}$ are given by $S_{\mathrm{b}\pm} = (S_{\mathrm{bx}} \pm iS_{\mathrm{by}})\, e^{\pm i\omega_{\mathrm{p}} t}$. The dipolar coupling term V is given by

$$V = g\hbar^{-1} (S_{\mathrm{a}+} + S_{\mathrm{a}-})\, S_{\mathrm{bz}}\ , \tag{6}$$

where g is a coupling rate.

First, the case $\Theta = 0$ is considered [see Eq. (2)]. Damping is taken into account by adding a Lindblad superoperator term $\mathcal{L}$ [51] to the master equation [see Eq. (A1) of appendix A]. The superoperator $\mathcal{L}$ depends on the thermal occupation factors $\hat{n}_{0\mathrm{a}}$ and $\hat{n}_{0\mathrm{b}}$, the longitudinal relaxation times $T_{1\mathrm{a}}$ and $T_{1\mathrm{b}}$, and the transverse relaxation times $T_{2\mathrm{a}}$ and $T_{2\mathrm{b}}$, of spins a and b, respectively. As can be seen from Eq. (A3) of appendix A, generally the Lindblad superoperator $\mathcal{L}$ linearly depends on the density matrix ρ, and consequently the master equation (A1) yields a unique steady state solution for the 4×4 complex and Hermitian matrix ρ (recall that in this section it is assumed that $\Theta = 0$). The corresponding 4×4 real Bloch matrix is denoted by B [see Eq. (C3) of appendix C]. The dependency of 15 (out of 16) matrix elements of B on driving detuning Δ and driving amplitude ω_1 is shown in Fig. 1 (the matrix element $B_{1,1}$, which is a constant by definition, is not shown). The plot in Fig. 1(b) displays the correlation parameter τ_{ab} [see Eq. (F2) of appendix F] as a function of driving parameters. As can be seen from the plots in Fig. 1, the largest effect of dipolar coupling occurs when the Hartmann–Hahn matching condition $\omega_{\mathrm{a}} = \omega_{\mathrm{R}}$ is satisfied, where $\omega_{\mathrm{R}} = \sqrt{\omega_1^2 + \Delta^2}$ is the Rabi angular frequency [55–57]. This matching condition is represented by the overlaid dashed white lines in the color coded plots in Fig. 1. Assumed parameters' values are listed in the caption of Fig. 1.

The first column of the matrix B yields spin a Bloch vector $\mathbf{k}_{\mathrm{a}} = (B_{2,1}, B_{3,1}, B_{4,1})$, whereas spin b Bloch vector $\mathbf{k}_{\mathrm{b}} = (B_{1,2}, B_{1,3}, B_{1,4})$ is extracted from the first row of B (recall that $B_{1,1}$ is a constant, see appendix C). The remaining 9 elements of the Bloch matrix B represent two–spin expectation values (e.g. $B_{4,4}$ is proportional to the expectation value $\langle S_{\mathrm{az}} S_{\mathrm{bz}}\rangle$).

For the case $g = 0$ (i.e. no dipolar coupling), in steady state the undriven spin a is in thermal equilibrium, and $\mathbf{k}_{\mathrm{a}} = (0, 0, k_{\mathrm{a},z})$, where $k_{\mathrm{a},z} = -1/(2\hat{n}_{0\mathrm{a}} + 1) = -\tanh(\beta\hbar\omega_{\mathrm{a}}/2)$ and $\beta = 1/(k_{\mathrm{B}} T)$ [see Eqs. (A4) and (A5) of appendix A]. For a finite coupling coefficient g, the undriven spin a in steady state is generally not in thermal equilibrium. However, an effective temperature T_{eff} can be defined based on the steady state value of $k_{\mathrm{a},z} = B_{4,1}$

$$T_{\mathrm{eff}} = \frac{\hbar\omega_{\mathrm{a}}}{2k_{\mathrm{B}} \tanh^{-1} k_{\mathrm{a},z}}\ . \tag{7}$$

The steady state value of $k_{\mathrm{a},z} = B_{4,1}$ is shown in Fig. 1 as a function of spin b driving detuning Δ and driving amplitude ω_1. The plot of $B_{4,1}$ (bottom left subplot in Fig. 1) reveals that $T_{\mathrm{eff}} > T$ for $\Delta < 0$, and $T_{\mathrm{eff}} < T$ for $\Delta > 0$, thus heating occurs with red–detuned driving (i.e. $\Delta < 0$) and cooling with blue–detuned driving (i.e. $\Delta > 0$). Note that similar driving–induced heating and cooling effects are observed with optomechanical cavities [58].

As can be seen from Fig. 1(b), when the Hartmann–Hahn matching condition $\omega_{\mathrm{a}} = \omega_{\mathrm{R}}$ is satisfied (see the overlaid white dashed line), the correlation parameter τ_{ab}, which quantifies the level of entanglement between the two spins (see appendix F), becomes relatively large.

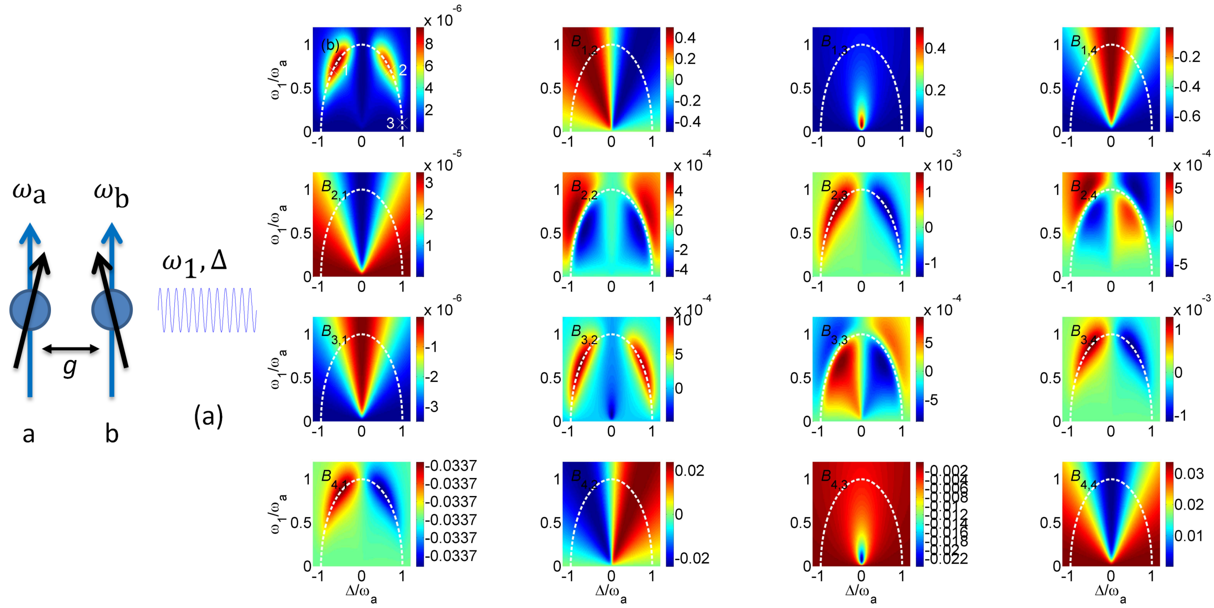

FIG. 1: Driving parameters. (a) A sketch of the two spin system. Steady state value of τ_{ab} is shown in (b), whereas the other color coded plots display the steady state value of 15 Bloch matrix B elements ($B_{1,1}$, which is a constant by definition, is not shown). Assumed parameters' values are $\gamma_{\mathrm{D}} = 0$ (i.e. no disentanglement), $g/\omega_{\mathrm{a}} = 10^{-3}$, $\Gamma_1^{(\mathrm{a})}/g = 10$, $\Gamma_\varphi^{(\mathrm{a})}/\Gamma_1^{(\mathrm{a})} = 10^{-4}$, $\Gamma_1^{(\mathrm{b})}/\Gamma_1^{(\mathrm{a})} = 10$, $\Gamma_\varphi^{(\mathrm{b})}/\Gamma_\varphi^{(\mathrm{a})} = 10$, $\hat{n}_0^{(\mathrm{a})} = 10$ and $\hat{n}_0^{(\mathrm{b})} = 10^{-4}$. The overlaid white $\times$ symbols in (b) represent the assumed driving parameters (detuning Δ and amplitude ω_1) for the plots shown in Fig. 2.

This observation suggests that the effect of disentanglement is expected to be relatively strong for $\omega_{\mathrm{a}} \simeq \omega_{\mathrm{R}}$.

III. DENSITY MATRIX DISENTANGLEMENT

The nonlinear term in the modified master equation (2) gives rise to disentanglement, provided that the expectation value $\langle \Theta \rangle$ of the ρ–dependent operator Θ quantifies the level of entanglement. The plots in Fig. 2 display time evolution of the single spin Bloch vectors $\mathbf{k}_{\mathrm{a}} = (B_{2,1}, B_{3,1}, B_{4,1})$ and $\mathbf{k}_{\mathrm{b}} = (B_{1,2}, B_{1,3}, B_{1,4})$, which are calculated by numerically integrating the modified master equation (2).

Two different methods to construct the operator Θ are employed for producing the plots in Fig. 2. For the plots labeled by the upper-case letter A, the operator Θ is given by $\Theta = \gamma_{\mathrm{D}} \mathcal{Q}_{\mathrm{ab}}^{(\mathrm{D})}$, whereas $\Theta = \gamma_{\mathrm{D}} Q_{\mathrm{a}}$ for plots labeled by the upper-case letter B. For both cases, the rate of disentanglement is denoted by γ_{D}. The operator $\mathcal{Q}_{\mathrm{ab}}^{(\mathrm{D})}$ [see Eq. (F1) of appendix F] gives rise to suppression of correlation between the two spins, whereas the operator Q_{a} [see Eq. (C7) of appendix C] generates Bloch matrix deranking. The labeling numbers 1, 2 and 3, which indicate the driving parameters $\Delta/\omega_{\mathrm{a}}$ and $\omega_1/\omega_{\mathrm{a}}$, refer to the overlaid white $\times$ symbols in Fig. 1(b). The lower-case letters a and b indicate the spin label. Assumed parameters' values are listed in the caption of Fig. 2.

As can be seen from Fig. 2, for both cases $\Theta = \gamma_{\mathrm{D}} \mathcal{Q}_{\mathrm{ab}}^{(\mathrm{D})}$ and $\Theta = \gamma_{\mathrm{D}} Q_{\mathrm{a}}$, with red–detuned driving the steady state is a fixed point [see the plots of Fig. 2 labeled by the number 1, and see Fig. 1(b)], whereas a limit cycle steady state can occur with blue–detuned driving [see the plots of Fig. 2 labeled by the numbers 2 and 3]. The plots in Fig. 2 also demonstrate that, even with the same value of the rate γ_{D}, the time evolutions generated by the operators $\mathcal{Q}_{\mathrm{ab}}^{(\mathrm{D})}$ and Q_{a} are clearly distinguishable.

IV. STATE VECTOR DISENTANGLEMENT

In the previous section, damping was taken into account using a deterministic master equation for the density operator ρ. Alternatively, damping can be accounted for using the stochastic Schrödinger–Langevin equation [52–54] for the state vector $|\psi\rangle$ [see Eq. (E1) of appendix E]. Disentanglement can be implemented by adding a Θ–dependent nonlinear term [see Eq. (1)].

The plots in Fig. 3 display time evolution of the single spin Bloch vectors $\mathbf{k}_{\mathrm{a}}$ and $\mathbf{k}_{\mathrm{b}}$, which are calculated by numerically integrating the modified Schrödinger–Langevin stochastic equation [which is constructed using Eqs. (1) and (E1)]. The rate of disentanglement γ_{D} is given by $\gamma_{\mathrm{D}}/\omega_{\mathrm{a}} = 0.1$ ($\gamma_{\mathrm{D}}/\omega_{\mathrm{a}} = 0.5$) for the plots labeled by upper-case letter A (B). The lower-case letters a and b indicate the spin label. Assumed parameters' values are listed in the caption of Fig. 3. The plots in both Figs. 2 and 3 demonstrate the richness of dynamical effects that

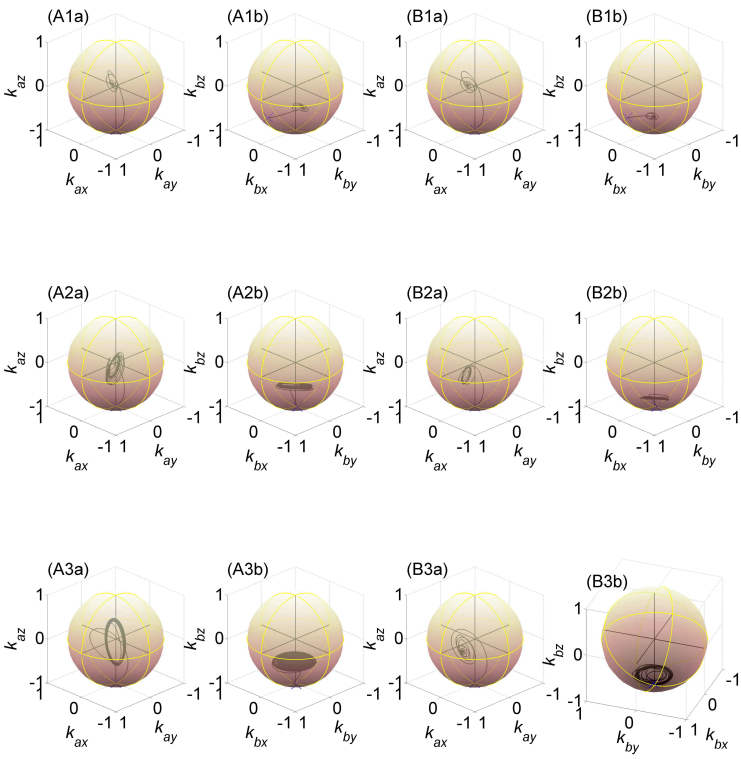

FIG. 2: Density matrix disentanglement. The capital letters (A and B) in the subplots' labeling indicate the method to construct the operator Θ (A for the case $\Theta = \gamma_{\mathrm{D}} \mathcal{Q}_{\mathrm{ab}}^{(\mathrm{D})}$, and B for the case $\Theta = \gamma_{\mathrm{D}} Q_{\mathrm{a}}$), the numbers (1, 2 and 3) indicate the driving parameters [see Fig. 1(b)], and the lower-case letters (a and b) indicate the spin label. Time evolution of the single spin Bloch vectors $\mathbf{k}_{\mathrm{a}}$ and $\mathbf{k}_{\mathrm{b}}$ is evaluated by numerically integrating the modified master equation (2). The blue $\times$ symbol represents initial state, which is determined from the steady state solution of the modified master equation (2) for the case $\Theta = 0$. Assumed parameters' values are $g/\omega_{\mathrm{a}} = 1$, $\Gamma_1^{(\mathrm{a})}/\omega_{\mathrm{a}} = 0.1$, $\Gamma_\varphi^{(\mathrm{a})}/\Gamma_1^{(\mathrm{a})} = 10^{-1}$, $\Gamma_1^{(\mathrm{b})}/\Gamma_1^{(\mathrm{a})} = 10$, $\Gamma_\varphi^{(\mathrm{b})}/\Gamma_\varphi^{(\mathrm{a})} = 10$, $\hat{n}_0^{(\mathrm{a})} = 5 \times 10^{-4}$, $\hat{n}_0^{(\mathrm{b})} = 1 \times 10^{-5}$ and $\gamma_{\mathrm{D}}/\omega_{\mathrm{a}} = 0.5$.

can be generated by models based on the spontaneous disentanglement hypothesis.

V. DISCUSSION

The results that are presented in Figs. 1, 2 and 3 are all based on numerical calculations. For sufficiently simple cases, analytical results, which can provide further insight, can be derived. In particular, for the two spin system under study in a pure state, the deranking operator Q_{S} [see Eq. (B10) of appendix B] is compared below to the correlation suppression operator $\mathcal{Q}_{\mathrm{ab}}^{(\mathrm{D})}$ [see Eq. (F1) of appendix F]. For a 2×2 state matrix M given by [see Eq. (B2) of appendix B]

$$M = \begin{pmatrix} \psi_1 & \psi_2 \\ \psi_3 & \psi_4 \end{pmatrix} , \tag{8}$$

the PSD matrix $\mathcal{G} = M M^\dagger$ is [see Eq. (B3) of appendix B]

$$\mathcal{G} = \begin{pmatrix} \psi_1\psi_1^* + \psi_2\psi_2^* & \psi_1\psi_3^* + \psi_2\psi_4^* \\ \psi_3\psi_1^* + \psi_4\psi_2^* & \psi_3\psi_3^* + \psi_4\psi_4^* \end{pmatrix} . \tag{9}$$

For this example $\det \mathcal{G} = |\det M|^2 = |\psi_1\psi_4 - \psi_2\psi_3|^2 = \delta/4$, where δ, which is defined by

$$\delta = 4 |\psi_1\psi_4 - \psi_2\psi_3|^2 , \tag{10}$$

is generally bounded by $\delta \in [0, 1]$. The eigenvalues of $\mathcal{G}$ are $(1/2)\left(1 \pm \sqrt{1-\delta}\right)$. Note that $\delta = 0$ ($\delta = 1$) for a fully disentangled (fully entangled) state. For the same pure state, the expectation value τ_{ab} of the correlation suppression operator $\mathcal{Q}_{\mathrm{ab}}^{(\mathrm{D})}$ is given by [see Eq. (F2) of appendix F]

$$\tau_{\mathrm{ab}} = \left\langle \mathcal{Q}_{\mathrm{ab}}^{(\mathrm{D})} \right\rangle = \frac{2\delta\left(1 + \frac{\delta}{2}\right)}{3} . \tag{11}$$

Thus, both the eigenvalues of $\mathcal{G}$ [which determine the deranking operator Q_{S}, see Eq. (B10) of appendix B]

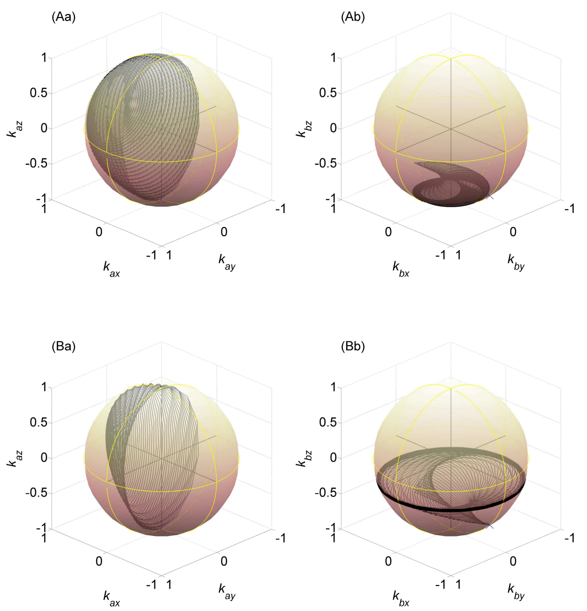

FIG. 3: State vector disentanglement. Time evolution of the single spin Bloch vectors $\mathbf{k}_a$ and $\mathbf{k}_b$ is evaluated by numerically integrating the modified Schrödinger–Langevin equation [see Eqs. (1) and (E1)]. Assumed parameters' values are $\Delta/\omega_a = \omega_1/\omega_a = 1/\sqrt{2}$ [these driving parameters correspond to the point labeled by the number 2 in Fig. 1(b)], $g/\omega_a = 100$, $\Gamma_1^{(a)}/\omega_a = 10^{-3}$, $\Gamma_\varphi^{(a)}/\Gamma_1^{(a)} = 0.1$, $\Gamma_1^{(b)}/\Gamma_1^{(a)} = 10$, $\Gamma_\varphi^{(b)}/\Gamma_\varphi^{(a)} = 10$, $\hat{n}_0^{(a)} = 5 \times 10^{-4}$ and $\hat{n}_0^{(b)} = 1 \times 10^{-5}$. For plots labeled by upper-case letters A and B, $\gamma_D/\omega_a = 0.1$ and $\gamma_D/\omega_a = 0.5$, respectively.

and τ_{ab} [see Eq. (11)] depend on δ [which is given by Eq. (10)]. On the other hand, the operator Q_S and $\mathcal{Q}_{ab}^{(D)}$ are generally not identical, and their impacts on dynamics are distinguishable.

The method of matrix deranking is being implemented in the current study to generate disentanglement in two different ways. In the first way, deranking is applied to the state matrix (see appendix B), whereas the Bloch matrix is being deranked in the second way (see appendix C). For the case of a pure state, Eq. (C9) of appendix C reveals that these two ways become effectively equivalent, provided that the two subsystems that are being disentangled share the same dimensionality.

The relation $\rho = \sum_i p_i |\alpha_i\rangle \langle\alpha_i|$ uniquely maps any given mixed state, which is characterized by probabilities $\{p_i\}$ (where $0 \leq p_i \leq 1$ and $\sum_i p_i = 1$), and corresponding normalized state vectors $\{|\alpha_i\rangle\}$, to a density operator ρ. On the other hand, a given ρ generally does not uniquely determine the mixed state (i.e. the probabilities $\{p_i\}$ and the corresponding normalized state vectors $\{|\alpha_i\rangle\}$). Nevertheless, in standard QM, all ensembles initially having the same ρ share the same time evolution, which is governed by a linear master equation for ρ. Moreover, in standard QM, the deterministic master equation, which governs the time evolution of ρ, is mathematically equivalent to the corresponding stochastic Schrödinger–Langevin equation for the state vector $|\psi\rangle$ [52]. As was discussed above, disentanglement is taken into account by adding a nonlinear term to the master equation. For a general mixed state, however, the impact of the nonlinear term on the time evolution of ρ is generally ensemble dependent, and it cannot be expressed as a function of ρ only. This observation suggests that, for the studying of the impact of disentanglement in the presence of environmental damping, it is usually advisable to implement a stochastic equation of motion for the state vector $|\psi\rangle$, rather than a deterministic master equation for the density operator ρ.

VI. SUMMARY

The current study explores some methods to implement the spontaneous disentanglement hypothesis. All methods are applicable for any physical system whose Hilbert space has finite dimensionality. Disentanglement has no effect on any product (i.e. disentangled) state, thus, all predictions of standard QM are unchanged in the absence of entanglement. The spontaneous disentanglement hypothesis is falsifiable - its predictions are distinguishable from what is obtained from standard QM. For a multipartite system, disentanglement between any pair of subsystems can be introduced. Disentanglement is invariant under any subsystem unitary transformation, and it is applicable for both distinguishable and indistinguishable particles [59]. Spontaneous disentanglement makes the collapse postulate of QM redundant.

All under–study methods to implement disentanglement require nonlinearity, because the subset of disentangled states in a Hilbert space of a given quantum composite system is generally not a subspace. Nonlinear effects [60–62], which are arguably inconsistent with standard QM [24], have been experimentally observed in a variety of small quantum systems [63–70]. Further study is needed to explore the possibility that the underlying mechanism responsible for the observed nonlinear effects is spontaneous disentanglement.

The process of disentanglement can give rise to limit cycle steady state solutions (see Figs. 2 and 3), which are otherwise theoretically excluded. Such limit cycle solutions occur above some threshold, which depends on the rate of disentanglement γ_D. Upper bounds upon γ_D can thus be derived from experiments studying driven spins. The results presented here provide some guidelines for experimentally testing the spontaneous disentanglement hypothesis. Even below the threshold, disentanglement has an impact. In particular, it affects the asymmetry in the response between red–detuned (i.e. $\Delta < 0$) and blue–detuned (i.e. $\Delta > 0$) driving. Moreover, disentanglement can give rise to multistability, which is otherwise excluded (for sufficiently small systems) [31, 71].

Appendix A: Damping

The GKSL master equation for the reduced density operator ρ is given by [16, 51, 72]

$$\frac{\mathrm{d}\rho}{\mathrm{d}t} = i\hbar^{-1}\left[\rho, \mathcal{H}\right] + \mathcal{L} \ , \tag{A1}$$

where $\mathcal{H} = \mathcal{H}^\dagger$ is the Hamiltonian, and $\mathcal{L}$ is a Lindblad superoperator [51]. For the two spin system under study, the coupling between spin L and its environment, where $\mathrm{L} \in \{\mathrm{a}, \mathrm{b}\}$, is characterized by energy–relaxation $\Gamma_1^{(\mathrm{L})}$ and dephasing $\Gamma_\varphi^{(\mathrm{L})}$ rates, thermal occupation factor $\hat{n}_0^{(\mathrm{L})}$, and longitudinal $T_1^{(\mathrm{L})}$ and transverse $T_2^{(\mathrm{L})}$ relaxation times. The Lindblad superoperator $\mathcal{L}$ is given by [73]

$$\begin{aligned}\mathcal{L} = \sum_{\mathrm{L}\in\{\mathrm{a},\mathrm{b}\}} & \frac{\left(\hat{n}_0^{(\mathrm{L})}+1\right)\Gamma_1^{(\mathrm{L})}}{4}\mathcal{D}_\rho\left(\frac{2S_{\mathrm{L},-}}{\hbar}\right) \\ & + \frac{\hat{n}_0^{(\mathrm{L})}\Gamma_1^{(\mathrm{L})}}{4}\mathcal{D}_\rho\left(\frac{2S_{\mathrm{L},+}}{\hbar}\right) \\ & + \frac{\left(2\hat{n}_0^{(\mathrm{L})}+1\right)\Gamma_\varphi^{(\mathrm{L})}}{2}\mathcal{D}_\rho\left(\frac{2S_{\mathrm{L},z}}{\hbar}\right) \ ,\end{aligned} \tag{A2}$$

where the Lindbladian $\mathcal{D}_\rho(X)$ for an operator X is given by

$$\mathcal{D}_\rho(X) = X\rho X^\dagger - \frac{X^\dagger X\rho + \rho X^\dagger X}{2} \ . \tag{A3}$$

The positive damping rates $\Gamma_1^{(\mathrm{L})}$ and $\Gamma_\varphi^{(\mathrm{L})}$, and the thermal occupation factor $\hat{n}_0^{(\mathrm{L})}$, are related to the longitudinal $T_1^{(\mathrm{L})}$ and the transverse $T_2^{(\mathrm{L})}$ relaxation times, and to the thermal equilibrium spin polarization $P_{z0}^{(\mathrm{L})}$, by $1/T_1^{(\mathrm{L})} = -\Gamma_1^{(\mathrm{L})}/P_{z0}^{(\mathrm{L})}$, $1/T_2^{(\mathrm{L})} = -\left(\Gamma_1^{(\mathrm{L})}/2 + \Gamma_\varphi^{(\mathrm{L})}\right)/P_{z0}^{(\mathrm{L})}$ and $-1/P_{z0}^{(\mathrm{L})} = 2\hat{n}_0^{(\mathrm{L})} + 1$.

As an example, consider a single spin 1/2 under transverse driving having amplitude ω_1 and angular frequency $\omega_\mathrm{p} = \omega_\mathrm{L} + \Delta$, where ω_L is the Larmor angular frequency, and Δ is the driving angular frequency detuning [see Eq. (5) for $\omega_\mathrm{a} = 0$ and $V = 0$]. For that case, the GKSL master equation (A1) yields a unique steady state solution, for which the expectation value of the spin angular momentum vector operator $\langle \mathbf{S}\rangle$ is given by [73]

$$\frac{2}{\hbar}\langle \mathbf{S}\rangle = \begin{pmatrix} \frac{\Delta\omega_1 T_2^2 P_{z0}}{1+\Delta^2T_2^2+\omega_1^2T_1T_2} \\ -\frac{\omega_1 T_2 P_{z0}}{1+\Delta^2T_2^2+\omega_1^2T_1T_2} \\ \frac{\left(1+\Delta^2T_2^2\right)P_{z0}}{1+\Delta^2T_2^2+\omega_1^2T_1T_2}\end{pmatrix} \ , \tag{A4}$$

where

$$P_{z0} = -\frac{1}{2\hat{n}+1} = -\tanh\frac{\beta\hbar\omega_\mathrm{L}}{2} \ , \tag{A5}$$

and $\beta = 1/(k_\mathrm{B}T)$.

Appendix B: State matrix and entanglement entropy

Consider a D_H–dimensional Hilbert space, where $D_\mathrm{H} \in \{4, 5, 6, \cdots\}$ is finite. Any state in the Hilbert space is represented by a complex $D_\mathrm{H}\times 1$ column vector given by

$$\left|\begin{matrix}\psi \\ D_\mathrm{H}\end{matrix}\right) = \begin{pmatrix}\psi_1 \\ \psi_2 \\ \vdots \\ \psi_{D_\mathrm{H}}\end{pmatrix} \ . \tag{B1}$$

The symbols $|)$ and $(|$ are henceforth used to denote column (ket) and row (bra) vectors, respectively. A unit $N\times 1$ column vector, whose m'th entry is given by $\delta_{m,n}$, where $n \in \{1, 2, \cdots, N\}$, is denoted by $\left|\begin{smallmatrix}n \\ N\end{smallmatrix}\right)$. For the case where the variable x does not represent an integer number, the symbol $\left|\begin{smallmatrix}x \\ N\end{smallmatrix}\right)$ denotes a general $N\times 1$ column vector, and the symbol $\left(\begin{smallmatrix}x \\ N\end{smallmatrix}\right|$ denotes its Hermitian conjugate $1\times N$ row vector.

Unless D_H is prime, it can be factored as $D_\mathrm{H} = D_\mathrm{a}D_\mathrm{b}$, where $D_\mathrm{a} > 1$ and $D_\mathrm{b} > 1$ are both integers. The two subsystems corresponding to the factorization [74] are labelled as 'a' and 'b', respectively. For any given factorization, the state ψ is represented by a $D_\mathrm{a}\times D_\mathrm{b}$ *state matrix* M given by

$$M = \begin{pmatrix} \psi_1 & \psi_2 & \cdots & \psi_{D_\mathrm{b}} \\ \psi_{D_\mathrm{b}+1} & \psi_{D_\mathrm{b}+2} & \cdots & \psi_{2D_\mathrm{b}} \\ \vdots & \vdots & \ddots & \vdots \\ \psi_{(D_\mathrm{a}-1)D_\mathrm{b}+1} & \psi_{(D_\mathrm{a}-1)D_\mathrm{b}+2} & & \psi_{D_\mathrm{a}D_\mathrm{b}}\end{pmatrix} \ . \tag{B2}$$

The $D_\mathrm{a}\times D_\mathrm{a}$ PSD matrix $\mathcal{G}$ is defined by

$$\mathcal{G} = MM^\dagger \ . \tag{B3}$$

For any eigenvalue λ of $\mathcal{G}$, the following holds $0 \leq \lambda \leq 1$, λ is an eigenvalue of the $D_\mathrm{b}\times D_\mathrm{b}$ PSD matrix $M^\dagger M$, and $\lambda^{1/2}$ is a singular value of M [75]. Note that the normalization condition $\left(\begin{smallmatrix}\psi \\ D_\mathrm{H}\end{smallmatrix}\middle|\begin{smallmatrix}\psi \\ D_\mathrm{H}\end{smallmatrix}\right) = 1$ yields $\operatorname{Tr}\mathcal{G} = 1$.

The level of bipartite entanglement (aka entanglement entropy) associated with a given pure state $\left|\begin{smallmatrix}\psi \\ D_\mathrm{H}\end{smallmatrix}\right)$ and a given factorization $D_\mathrm{H} = D_\mathrm{a}D_\mathrm{b}$, which is henceforth denoted by $\mathcal{K}$, can be characterized in term of the normalized rank $\mathcal{R}(\mathcal{G})$ of the $D_\mathrm{a}\times D_\mathrm{a}$ PSD matrix $\mathcal{G}$ as [recall that $\operatorname{Tr}\mathcal{G} = 1$, and see Eq. (4)]

$$\mathcal{K} = \log(D_\mathrm{a})\,\mathcal{R}(\mathcal{G}) = \operatorname{Tr}(-\mathcal{G}\log\mathcal{G}) \ . \tag{B4}$$

Note that $\mathcal{K}$ is bounded by $\mathcal{K} \in [0, \log(\min\{D_\mathrm{a}, D_\mathrm{b}\})]$. To implement disentanglement by deranking, the level of bipartite entanglement $\mathcal{K}$ has to be expressed as an expectation value. An Hermitian operator Q_S satisfying the relation $\langle Q_\mathrm{S}\rangle = \mathcal{K}$ is derived below.

The state matrix M can be expressed as

$$M = \sum_{d_\mathrm{a}=1}^{D_\mathrm{a}}\sum_{d_\mathrm{b}=1}^{D_\mathrm{b}} \mu_{d_\mathrm{a},D_\mathrm{a},d_\mathrm{b},D_\mathrm{b}}\left|\begin{matrix}\psi \\ D_\mathrm{a}D_\mathrm{b}\end{matrix}\right)\left(\begin{matrix}d_\mathrm{b} \\ D_\mathrm{b}\end{matrix}\right| \ , \tag{B5}$$

where the $D_{\rm a}\times(D_{\rm a}D_{\rm b})$ matrix $\mu_{d_{\rm a},D_{\rm a},d_{\rm b},D_{\rm b}}$ is given by

$$\begin{aligned}\mu_{d_{\rm a},D_{\rm a},d_{\rm b},D_{\rm b}} &= \left|\begin{matrix}d_{\rm a}\\D_{\rm a}\end{matrix}\right)\left(\left(\begin{matrix}d_{\rm a}\\D_{\rm a}\end{matrix}\right|\otimes\left(\begin{matrix}d_{\rm b}\\D_{\rm b}\end{matrix}\right|\right)\\ &= \left|\begin{matrix}d_{\rm a}\\D_{\rm a}\end{matrix}\right)\left(\begin{matrix}(d_{\rm a}-1)\,D_{\rm b}+d_{\rm b}\\D_{\rm a}D_{\rm b}\end{matrix}\right| ,\end{aligned} \tag{B6}$$

and where the symbol $\otimes$ denotes Kronecker matrix product, and thus

$$\mathcal{G}\equiv MM^{\dagger}=\sum_{d_{\rm b}=1}^{D_{\rm b}}\sum_{d'_{\rm a}=1}^{D_{\rm a}}\sum_{d''_{\rm a}=1}^{D_{\rm a}}\mu_{d'_{\rm a},D_{\rm a},d_{\rm b},D_{\rm b}}\rho_{\psi}\mu^{\dagger}_{d''_{\rm a},D_{\rm a},d_{\rm b},D_{\rm b}} , \tag{B7}$$

where

$$\rho_{\psi}=\left|\begin{matrix}\psi\\D_{\rm a}D_{\rm b}\end{matrix}\right)\left(\begin{matrix}\psi\\D_{\rm a}D_{\rm b}\end{matrix}\right| . \tag{B8}$$

The definition of the level of entanglement $\mathcal{K}$, which for a pure state is given by Eq. (B4), is generalized for a mixed state by [see Eq. (B7), and recall the identity $\mathrm{Tr}\left(|u\rangle\langle v|\right)=\langle v\,|u\rangle$]

$$\mathcal{K}=\mathrm{Tr}\left(\rho Q_{\rm S}\right)\equiv\langle Q_{\rm S}\rangle , \tag{B9}$$

where the operator $Q_{\rm S}$ is given by

$$Q_{\rm S}=-\sum_{d_{\rm b}=1}^{D_{\rm b}}\sum_{d'_{\rm a}=1}^{D_{\rm a}}\sum_{d''_{\rm a}=1}^{D_{\rm a}}\mu^{\dagger}_{d''_{\rm a},D_{\rm a},d_{\rm b},D_{\rm b}}\left(\log\mathcal{G}\right)\mu_{d'_{\rm a},D_{\rm a},d_{\rm b},D_{\rm b}} , \tag{B10}$$

and where ρ is the density matrix of the given state [the matrix $\mathcal{G}$ is calculated using Eq. (B7) for $\rho_{\psi}=\rho$].

Appendix C: Bloch matrix

For a given Hilbert space having dimensionality $D_{\rm H}$, and for a given factorization $D_{\rm H}=D_{\rm a}D_{\rm b}$, where $D_{\rm a}>1$ and $D_{\rm b}>1$ are both integers, a state matrix M [see Eq. (B2) of appendix B] can be defined for any *pure* state, whereas a Bloch matrix can be defined for any general (i.e. mixed) state. The generalized Gell-Mann set $\{\lambda_l\}$, which spans the SU($D_{\rm H}$) Lie algebra, contains $D_{\rm H}^2-1$ square $D_{\rm H}\times D_{\rm H}$ Hermitian matrices. For the case $D_{\rm H}=2$ ($D_{\rm H}=3$), the $D_{\rm H}^2-1=3$ ($D_{\rm H}^2-1=8$) elements are called Pauli (Gell-Mann) matrices. The Generalized Gell-Mann matrices are traceless, i.e. $\mathrm{Tr}\,\lambda_l=0$, and they satisfy the orthogonality relation

$$\frac{\mathrm{Tr}\left(\lambda_{l'}\lambda_{l''}\right)}{2}=\delta_{l',l''} . \tag{C1}$$

For a given factorization $D_{\rm H}=D_{\rm a}D_{\rm b}$, the generalized Gell-Mann $D_{\rm L}\times D_{\rm L}$ matrices corresponding to subsystem L, where $\mathrm{L}\in\{\mathrm{a},\mathrm{b}\}$, are denoted by $\lambda_l^{(\mathrm{L})}$, where $l\in\left\{1,2,\cdots,D_{\rm L}^2-1\right\}$. Consider the set of $D_{\rm H}^2-1$ matrices $G^{(\mathrm{ab})}=\left\{\Gamma_a^{(\mathrm{a})}\otimes\Gamma_b^{(\mathrm{b})}\right\}-\left\{\Gamma_0^{(\mathrm{a})}\otimes\Gamma_0^{(\mathrm{b})}\right\}$, where $a\in\left\{0,1,2,\cdots,D_{\rm a}^2-1\right\}$ and $b\in\left\{0,1,2,\cdots,D_{\rm b}^2-1\right\}$. For subsystem L, where $\mathrm{L}\in\{\mathrm{a},\mathrm{b}\}$, the matrix $\Gamma_0^{(\mathrm{L})}$ is defined by $\Gamma_0^{(\mathrm{L})}=\left(2^{1/4}/D_{\rm L}^{1/2}\right)I_{\rm L}$, where $I_{\rm L}$ is the $D_{\rm L}\times D_{\rm L}$ identity matrix, and for $l\in\left\{1,2,\cdots,D_{\rm L}^2-1\right\}$ the matrix $\Gamma_l^{(\mathrm{L})}$ is defined by $\Gamma_l^{(\mathrm{L})}=2^{-1/4}\lambda_l^{(\mathrm{L})}$.

With the help of the Kronecker matrix product identities $\mathrm{Tr}\left(X_1\otimes X_2\right)=\mathrm{Tr}\,X_1\,\mathrm{Tr}\,X_2$ and $\left(X_1\otimes X_2\right)\left(X_3\otimes X_4\right)=\left(X_1X_3\right)\otimes\left(X_2X_4\right)$, one finds that the set $G^{(\mathrm{ab})}$ shares two properties with the Gell-Mann set G of the $D_{\rm H}$-dimensional Hilbert space. The first one is tracelessness $\mathrm{Tr}\,G_{a,b}=0$ for any $G_{a,b}\equiv\Gamma_a^{(\mathrm{a})}\otimes\Gamma_b^{(\mathrm{b})}\in G^{(\mathrm{ab})}$ [recall that $G_{0,0}\notin G^{(\mathrm{ab})}$], and the second one is orthogonality [see Eq. (C1)]

$$\frac{\mathrm{Tr}\left(G_{a',b'}G_{a'',b''}\right)}{2}=\delta_{a',a''}\delta_{b',b''} . \tag{C2}$$

The $D_{\rm a}^2\times D_{\rm b}^2$ matrix B, where

$$B_{a,b}=\langle G_{a,b}\rangle , \tag{C3}$$

is henceforth referred to as the Bloch matrix. The following holds $B_{0,0}=\sqrt{2/\left(D_{\rm a}D_{\rm b}\right)}$, and $\mathrm{Tr}\left(BB^{\dagger}\right)=\mathrm{Tr}\left(B^{\dagger}B\right)=2\,\mathrm{Tr}\,\rho^2$ [see Eq. (C2), and recall the identity $\mathrm{Tr}\left(X_1\otimes X_2\right)=\mathrm{Tr}\,X_1\,\mathrm{Tr}\,X_2$]. Expectation value $\langle A\rangle=\mathrm{Tr}\left(\rho A\right)$ of a given observable A is given by

$$\langle A\rangle=\sum_{a=0}^{D_{\rm a}^2-1}\sum_{b=0}^{D_{\rm b}^2-1}\frac{\mathrm{Tr}\left(\rho G_{a,b}\right)\mathrm{Tr}\left(AG_{a,b}\right)}{2} . \tag{C4}$$

The Bloch matrix B can be used to define an alternative quantification for the level of bipartite entanglement $\mathcal{L}$, which is given by [compare with Eq. (B4)]

$$\mathcal{L}=-\,\mathrm{Tr}\left(\alpha\log\alpha\right)=-\,\mathrm{Tr}\left(\beta\log\beta\right) , \tag{C5}$$

where the $D_{\rm a}^2\times D_{\rm a}^2$ matrix α is given by $\alpha=(1/2)\,BB^{\dagger}$, and the $D_{\rm b}^2\times D_{\rm b}^2$ matrix β is given by $\beta=(1/2)\,B^{\dagger}B$ (recall that the PSD matrices α and β share the same set of eigenvalues, and the same trace $\mathrm{Tr}\,\alpha=\mathrm{Tr}\,\beta=\mathrm{Tr}\,\rho^2$). The following holds [see Eq. (C5)]

$$\langle Q_{\rm a}\rangle=\langle Q_{\rm b}\rangle=\mathcal{L} , \tag{C6}$$

where the operators $Q_{\rm a}$ and $Q_{\rm b}$ are defined by

$$Q_{\rm a}=\mathrm{Tr}\left(-\frac{GB^{\dagger}}{2}\log\left(\frac{BB^{\dagger}}{2}\right)\right) , \tag{C7}$$

$$Q_{\rm b}=\mathrm{Tr}\left(-\log\left(\frac{B^{\dagger}B}{2}\right)\frac{B^{\dagger}G}{2}\right) . \tag{C8}$$

The (a,b) entry of the $D_{\rm a}^2\times D_{\rm b}^2$ matrix G is the $D_{\rm H}\times D_{\rm H}$ observable $G_{a,b}=\Gamma_a^{(\mathrm{a})}\otimes\Gamma_b^{(\mathrm{b})}$, and the (a,b) entry of the $D_{\rm a}^2\times D_{\rm b}^2$ matrix B is the expectation value $B_{a,b}=\langle G_{a,b}\rangle$ [note that $a\in\left\{0,1,\cdots,D_{\rm a}^2-1\right\}$ and

$b \in \{0, 1, \cdots, D_{\mathrm{b}}^2 - 1\}$]. As can be see from Eq. (C6), the operators Q_{a} and Q_{b} can be used for the implementation of disentanglement that is based on the deranking of the matrices α and β.

Consider a pure state that is characterized by a state matrix M [see Eq. (B2) of appendix B] and a Bloch matrix B [see Eq. (C3)]. Recall that for a pure state $\mathrm{Tr}\left((1/2) BB^\dagger\right) = \mathrm{Tr}\left((1/2) B^\dagger B\right) = \mathrm{Tr}\,\rho^2 = 1$. For the case $D_{\mathrm{a}} = D_{\mathrm{b}}$, the Hermitian matrices $\left(M^\dagger M\right) \otimes \left(M^\dagger M\right)$ and $(1/2) BB^\dagger$ are unitarily equivalent [proof is based on Specht's theorem, see Eq. (8.1046) of Ref. [30]]. This implies that for this case the entanglement parameter $\mathcal{K}$ [see Eq. (B4) of appendix B], which is based on the state matrix M [see Eq. (B2) of appendix B], and the entanglement parameter $\mathcal{L}$ [see Eq. (C5)], which is based on the Bloch matrix B (C3), are related by [recall the tensor product identity $\mathrm{Tr}\,(X \otimes Y) = \mathrm{Tr}\,X\,\mathrm{Tr}\,Y$]

$$2\mathcal{K} = \mathcal{L}\,. \tag{C9}$$

Appendix D: Weyl basis

The generalized Gell-Mann matrices have been employed in appendix C to span the SU(D_{H}) Lie algebra. Alternatively, the Weyl operators can be used for the same purpose [76–78]. Let $\{|n\rangle\}$, where $n \in \{0, 1, 2, \cdots, D-1\}$, be an orthonormal basis for a Hilbert space of a D level system. The Weyl operator $W_{n'n''}$ is defined by

$$W_{n'n''} = \sum_{n=0}^{D-1} e^{\frac{2\pi i}{D} nn'} |n\rangle \langle n + n''|\ , \tag{D1}$$

where $n', n'' \in \{0, 1, 2, \cdots, D-1\}$. For any integer n, the abbreviated notation $|n\rangle$ denotes the state $|\mathrm{mod}\,(n, D)\rangle$. For a mixed state represented by a $D \times D$ density matrix ρ, the elements of the Weyl matrix $\mathcal{W}$ are given by (note that elements' numbering starts from zero)

$$\mathcal{W}_{n'n''} = \frac{\mathrm{Tr}\,(W_{n'n''}\rho)}{\sqrt{D}}\ , \tag{D2}$$

where $n', n'' \in \{0, 1, 2, \cdots, D-1\}$. For a pure state $\mathrm{Tr}\left(\mathcal{W}^\dagger \mathcal{W}\right) = 1$ and $1/D \leq \mathrm{Tr}\left(\left(\mathcal{W}^\dagger \mathcal{W}\right)^2\right) \leq 1$, whereas for a mixed state $1/D^2 \leq \mathrm{Tr}\left(\mathcal{W}^\dagger \mathcal{W}\right) = \mathrm{Tr}\,\rho^2 \leq 1$.

Consider a system composed of two subsystems labelled by the letters a and b, respectively. The Hilbert space dimensionality of subsystem a (b) is denoted by D_{a} (D_{b}). The Weyl operators (D1) corresponding to subsystem L, where $\mathrm{L} \in \{\mathrm{a}, \mathrm{b}\}$, are denoted by $W^{(\mathrm{L})}_{n'n''}$, where $n', n'' \in \{0, 1, 2, \cdots, D_{\mathrm{L}} - 1\}$. For a given $D_{\mathrm{a}}D_{\mathrm{b}} \times D_{\mathrm{a}}D_{\mathrm{b}}$ density matrix ρ, the elements of the $D_{\mathrm{a}}D_{\mathrm{b}} \times D_{\mathrm{a}}D_{\mathrm{b}}$ Weyl matrix $\mathcal{W}$ are given by [see Eq. (D2)]

$$\mathcal{W}_{(n',n'''),(n'',n'''')} = \frac{\mathrm{Tr}\left(\left(W^{(\mathrm{a})}_{n'n''} \otimes W^{(\mathrm{b})}_{n'''n''''}\right)\rho\right)}{\sqrt{D_{\mathrm{a}}D_{\mathrm{b}}}}\ , \tag{D3}$$

where $n', n'' \in \{0, 1, 2, \cdots, D_{\mathrm{a}} - 1\}$ and $n''', n'''' \in \{0, 1, 2, \cdots, D_{\mathrm{b}} - 1\}$. For a product state having a density matrix ρ given by

$$\rho = \rho^{(\mathrm{a})} \otimes \rho^{(\mathrm{b})}\ , \tag{D4}$$

the elements of the $D_{\mathrm{a}}D_{\mathrm{b}} \times D_{\mathrm{a}}D_{\mathrm{b}}$ Weyl matrix $\mathcal{W}$ (D3) are given by [see Eq. (8.242) of Ref. [30]]

$$\mathcal{W}_{(n',n'''),(n'',n'''')} = \mathcal{W}^{(\mathrm{a})}_{n'n''} \mathcal{W}^{(\mathrm{b})}_{n'''n''''}\ , \tag{D5}$$

where $\mathcal{W}^{(\mathrm{a})}$ and $\mathcal{W}^{(\mathrm{b})}$ are the Weyl matrices of subsystems a and b, respectively.

Consider a pure state of the composed system given by $|\psi\rangle \doteq \left(\psi_1, \psi_2, \cdots, \psi_{D_{\mathrm{a}}D_{\mathrm{b}}}\right)^{\mathrm{T}}$. The state vector $|\psi\rangle$ is represented by a $D_{\mathrm{a}} \times D_{\mathrm{b}}$ state matrix M given by Eq. (B2) of appendix B. For the case where $D_{\mathrm{a}} = D_{\mathrm{b}} \equiv D$, the $D^2 \times D^2$ Weyl $\mathcal{S}$ matrix is defined by

$$\mathcal{S}_{n'+n''D, n'''+n''''D} = \frac{1}{D} \mathrm{Tr}\,\left(\left(W_{n',n''} \otimes W_{n''',n''''}\right)\rho\right)\ , \tag{D6}$$

where $n', n'', n''', n'''' \in \{0, 1, 2, \cdots, D-1\}$, and ρ is the system's density matrix. The PSD matrices $\mathcal{M} = \left(M^\dagger M\right) \otimes \left(M^\dagger M\right)$ and $\mathcal{S}^\dagger \mathcal{S}$ share the same trace $\mathrm{Tr}\,\mathcal{M} = \mathrm{Tr}\left(\mathcal{S}^\dagger \mathcal{S}\right) = 1$, and the same eigenvalues (i.e. they are unitarily equivalent) [see Eq. (8.245) of Ref. [30]]. The parameter $\mathrm{Tr}\left(\left(\mathcal{S}^\dagger \mathcal{S}\right)^2\right)$ can be used to quantify entanglement of a pure state $|\psi\rangle$, for a system composed of two subsystems, each having Hilbert space dimensionality D. For any fully disentangled state $\mathrm{Tr}\left(\left(\mathcal{S}^\dagger \mathcal{S}\right)^2\right) = 1$, whereas $\mathrm{Tr}\left(\left(\mathcal{S}^\dagger \mathcal{S}\right)^2\right) = 1/D^2$ for any fully entangled state. For the implementation of disentanglement using the modified Schrödinger equation (1), the following relation can be used

$$\mathrm{Tr}\left(\left(\mathcal{S}^\dagger \mathcal{S}\right)^2\right) = \langle\psi| T_2 |\psi\rangle\ , \tag{D7}$$

where the operator T_2 is given by

$$\begin{aligned} T_2 = \frac{1}{D^4} \sum_{n_1, n_2, \cdots, n_8 = 0}^{D-1} & \left(W^\dagger_{n_3,n_4} \otimes W^\dagger_{n_1,n_2}\right)\rho \\ & \times \left(W_{n_3,n_4} \otimes W_{n_5,n_6}\right)\rho \\ & \times \left(W^\dagger_{n_7,n_8} \otimes W^\dagger_{n_5,n_6}\right)\rho \\ & \times \left(W_{n_7,n_8} \otimes W_{n_1,n_2}\right)\ , \end{aligned} \tag{D8}$$

and $\rho = |\psi\rangle \langle\psi|$ is the $N^2 \times N^2$ density matrix associated with the pure state $|\psi\rangle$.

Appendix E: The Schrödinger–Langevin equation

While the GKSL master equation (A1) of appendix A, which governs the time evolution of the reduced density matrix ρ, is deterministic, the effect of damping on the time evolution of the state vector $|\psi\rangle$ can be accounted for by a stochastic equation of motion, which is known as the Schrödinger–Langevin equation, and which is given by [52–54]

$$\frac{\mathrm{d}\,|\psi\rangle}{\mathrm{d}t} = \left[-i\hbar^{-1}\mathcal{H} + \sum_l \left(\xi_l\,(t)\,\mathcal{V}_l - \frac{1}{2}\mathcal{V}_l^\dagger\mathcal{V}_l\right)\right]|\psi\rangle \ . \tag{E1}$$

The random functions of time $\xi_l\,(t)$, which represent white noise, have vanishing averaged values $\overline{\xi_l\,(t)} = 0$, and correlation functions given by

$$\overline{\xi_{l'}\,(t')\,\xi^*_{l''}\,(t'')} = \delta_{l',l''}\delta\,(t'-t'') \ , \tag{E2}$$

where overbar denotes time averaging. For the under study two spin system, the time independent operators $\mathcal{V}_l$ for spin L are $\hbar^{-1}\sqrt{\left(\hat{n}_0^{(\mathrm{L})}+1\right)\Gamma_1^{(\mathrm{L})}}S_{\mathrm{L},-}$, $\hbar^{-1}\sqrt{\hat{n}_0^{(\mathrm{L})}\Gamma_1^{(\mathrm{L})}}S_{\mathrm{L},+}$ and $\hbar^{-1}\sqrt{2\left(2\hat{n}_0^{(\mathrm{L})}+1\right)\Gamma_\varphi^{(\mathrm{L})}}S_{\mathrm{L},z}$, where $\mathrm{L}\in\{\mathrm{a},\mathrm{b}\}$ [see Eq. (A3) of appendix A].

Appendix F: Correlation suppression

Consider a multipartite system composed of three subsystems labeled as 'a', 'b' and 'c'. The Hilbert space of the system $H = H_\mathrm{a}\otimes H_\mathrm{b}\otimes H_\mathrm{c}$ is a tensor product of subsystem Hilbert spaces H_a, H_b and H_c. The dimensionality of the Hilbert space H_L of subsystem L, which is denoted by D_L, where $\mathrm{L}\in\{\mathrm{a},\mathrm{b},\mathrm{c}\}$, is assumed to be finite. The PSD operator $\mathcal{Q}_{\mathrm{ab}}^{(\mathrm{D})}$, which is defined below in Eq. (F1), can be used in both the modified Schrödinger equation (1), and in the modified master equation (2), to suppress correlation between subsystems a and b.

Let $\left\{\lambda_1^{(\mathrm{L})},\lambda_2^{(\mathrm{L})},\cdots,\lambda_{D_\mathrm{L}^2-1}^{(\mathrm{L})}\right\}$ be a basis spanning the SU(D_L) Lie algebra corresponding to subsystem L, where $\mathrm{L}\in\{\mathrm{a},\mathrm{b},\mathrm{c}\}$. In the current study, both generalized Gell-Mann (see appendix C) and Weyl (see appendix D) bases are used for the construction of the decorrelating operator $\mathcal{Q}^{(\mathrm{D})}$, which is given by

$$\mathcal{Q}_{\mathrm{ab}}^{(\mathrm{D})} = \eta_{\mathrm{ab}}\,\mathrm{Tr}\left(C^{\mathrm{T}}\,\langle C\rangle\right) \ . \tag{F1}$$

The (a,b) entry of the $\left(D_\mathrm{a}^2-1\right)\times\left(D_\mathrm{b}^2-1\right)$ matrix C is the observable $\mathcal{C}\left(\lambda_a^{(\mathrm{a})},\lambda_b^{(\mathrm{b})}\right)$, and the (a,b) entry of the $\left(D_\mathrm{a}^2-1\right)\times\left(D_\mathrm{b}^2-1\right)$ matrix $\langle C\rangle$ is its expectation value $\left\langle\mathcal{C}\left(\lambda_a^{(\mathrm{a})},\lambda_b^{(\mathrm{b})}\right)\right\rangle$. For any given observable $O_\mathrm{a} = O_\mathrm{a}^\dagger$ of subsystem a, and a given observable $O_\mathrm{b} = O_\mathrm{b}^\dagger$ of subsystem b, the observable $\mathcal{C}\,(O_\mathrm{a},O_\mathrm{b})$ is defined by $\mathcal{C}\,(O_\mathrm{a},O_\mathrm{b}) = O_\mathrm{a}\otimes O_\mathrm{b}\otimes I_\mathrm{c} - \langle O_\mathrm{a}\otimes I_\mathrm{b}\otimes I_\mathrm{c}\rangle\,\langle I_\mathrm{a}\otimes O_\mathrm{b}\otimes I_\mathrm{c}\rangle$, where I_L is the $D_\mathrm{L}\times D_\mathrm{L}$ identity matrix, and where $\mathrm{L}\in\{\mathrm{a},\mathrm{b},\mathrm{c}\}$. Note that the nonnegative expectation value τ_{ab}, which is given by

$$\tau_{\mathrm{ab}} = \left\langle\mathcal{Q}_{\mathrm{ab}}^{(\mathrm{D})}\right\rangle \ , \tag{F2}$$

is invariant under any single subsystem unitary transformation. The positive constant η_{ab} [see Eq. (F1)] is chosen such that the expectation value τ_{ab} is generally bounded by $\tau_{\mathrm{ab}}\in[0,1]$. For the two spin 1/2 system under study $\eta_{\mathrm{ab}} = 1/3$.

[1] Steven Weinberg, "Precision tests of quantum mechanics", in *THE OSKAR KLEIN MEMORIAL LECTURES 1988–1999*, pp. 61–68. World Scientific, 2014.

[2] H-D Doebner and Gerald A Goldin, "Introducing nonlinear gauge transformations in a family of nonlinear schrödinger equations", *Physical Review A*, vol. 54, no. 5, pp. 3764, 1996.

[3] Nicolas Gisin and Ian C Percival, "The quantum-state diffusion model applied to open systems", *Journal of Physics A: Mathematical and General*, vol. 25, no. 21, pp. 5677, 1992.

[4] Nicolas Gisin, "A simple nonlinear dissipative quantum evolution equation", *Journal of Physics A: Mathematical and General*, vol. 14, no. 9, pp. 2259, 1981.

[5] David E Kaplan and Surjeet Rajendran, "Causal framework for nonlinear quantum mechanics", *Physical Review D*, vol. 105, no. 5, pp. 055002, 2022.

[6] Manuel H Muñoz-Arias, Pablo M Poggi, Poul S Jessen, and Ivan H Deutsch, "Simulating nonlinear dynamics of collective spins via quantum measurement and feedback", *Physical review letters*, vol. 124, no. 11, pp. 110503, 2020.

[7] Michael R Geller, "Fast quantum state discrimination with nonlinear positive trace-preserving channels", *Advanced Quantum Technologies*, p. 2200156, 2023.

[8] Gian Carlo Ghirardi, Alberto Rimini, and Tullio Weber, "Unified dynamics for microscopic and macroscopic systems", *Physical review D*, vol. 34, no. 2, pp. 470, 1986.

[9] E. Schrodinger, "Die gegenwartige situation in der quantenmechanik", *Naturwissenschaften*, vol. 23, pp. 807, 1935.

[10] Roger Penrose, "Uncertainty in quantum mechanics: faith or fantasy?", *Philosophical Transactions of the Royal Society A: Mathematical, Physical and Engineering Sciences*, vol. 369, no. 1956, pp. 4864–4890, 2011.

[11] Jonathan Oppenheim, "A postquantum theory of classical gravity?", *Physical Review X*, vol. 13, no. 4, pp. 041040, 2023.

[12] Angelo Bassi, Kinjalk Lochan, Seema Satin, Tejinder P Singh, and Hendrik Ulbricht, "Models of wave-function collapse, underlying theories, and experimental tests", *Reviews of Modern Physics*, vol. 85, no. 2, pp. 471, 2013.

[13] Philip Pearle, "Reduction of the state vector by a nonlinear schrödinger equation", *Physical Review D*, vol. 13, no. 4, pp. 857, 1976.

[14] Angelo Bassi and GianCarlo Ghirardi, "Dynamical reduction models", *Physics Reports*, vol. 379, no. 5-6, pp. 257–426, 2003.

[15] Krzysztof Kowalski, "Linear and integrable nonlinear

evolution of the qutrit", *Quantum Information Processing*, vol. 19, no. 5, pp. 1–31, 2020.

[16] Bernd Fernengel and Barbara Drossel, "Bifurcations and chaos in nonlinear lindblad equations", *Journal of Physics A: Mathematical and Theoretical*, vol. 53, no. 38, pp. 385701, 2020.

[17] Philippe Chomaz and Francesca Gulminelli, "Phase transitions in finite systems", in *Dynamics and thermodynamics of systems with long-range interactions*, pp. 68–129. Springer, 2002.

[18] Paul Mainwood, "Phase transitions in finite systems", 2005.

[19] Craig Callender, "Taking thermodynamics too seriously", *Studies in history and philosophy of science part B: studies in history and philosophy of modern physics*, vol. 32, no. 4, pp. 539–553, 2001.

[20] Chuang Liu, "Explaining the emergence of cooperative phenomena", *Philosophy of Science*, vol. 66, no. S3, pp. S92–S106, 1999.

[21] Vincent Ardourel and Sorin Bangu, "Finite-size scaling theory: Quantitative and qualitative approaches to critical phenomena", *Studies in History and Philosophy of Science*, vol. 100, pp. 99–106, 2023.

[22] Elay Shech, "What is the paradox of phase transitions?", *Philosophy of Science*, vol. 80, no. 5, pp. 1170–1181, 2013.

[23] Eyal Buks, "Spontaneous disentanglement and thermalization", *Advanced Quantum Technologies*, vol. 7, no. 5, pp. 2400036, 2024.

[24] Eyal Buks, "Disentanglement-induced multistability", *Physical Review A*, vol. 110, no. 1, pp. 012439, 2024.

[25] R Grimaudo, Asm De Castro, M Kuś, and A Messina, "Exactly solvable time-dependent pseudo-hermitian su (1, 1) hamiltonian models", *Physical Review A*, vol. 98, no. 3, pp. 033835, 2018.

[26] K Kowalski and J Rembieliński, "Integrable nonlinear evolution of the qubit", *Annals of Physics*, vol. 411, pp. 167955, 2019.

[27] Andreas Elben, Richard Kueng, Hsin-Yuan Robert Huang, Rick van Bijnen, Christian Kokail, Marcello Dalmonte, Pasquale Calabrese, Barbara Kraus, John Preskill, Peter Zoller, et al., "Mixed-state entanglement from local randomized measurements", *Physical Review Letters*, vol. 125, no. 20, pp. 200501, 2020.

[28] Alessandro Sergi and Konstantin G Zloshchastiev, "Non-hermitian quantum dynamics of a two-level system and models of dissipative environments", *International Journal of Modern Physics B*, vol. 27, no. 27, pp. 1350163, 2013.

[29] Dorje C Brody and Eva-Maria Graefe, "Mixed-state evolution in the presence of gain and loss", *Physical review letters*, vol. 109, no. 23, pp. 230405, 2012.

[30] Eyal Buks, *Quantum mechanics - Lecture Notes*, http://buks.net.technion.ac.il/teaching/, 2025.

[31] Eyal Buks, "Disentanglement–induced bistability in a magnetic resonator", *Advanced Quantum Technologies*, p. 2400587, 2025.

[32] Ruben Daraban, Fabrizio Salas-Ramírez, and Johannes Schachenmayer, "Non-unitarity maximizing unraveling of open quantum dynamics", *SciPost Physics*, vol. 18, no. 2, pp. 048, 2025.

[33] H Grabert, "Nonlinear relaxation and fluctuations of damped quantum systems", *Zeitschrift für Physik B Condensed Matter*, vol. 49, no. 2, pp. 161–172, 1982.

[34] Hans Christian Öttinger, "Nonlinear thermodynamic quantum master equation: Properties and examples", *Physical Review A*, vol. 82, no. 5, pp. 052119, 2010.

[35] Olivier Roy and Martin Vetterli, "The effective rank: A measure of effective dimensionality", in *2007 15th European signal processing conference*. IEEE, 2007, pp. 606–610.

[36] Jürgen Schlienz and Günter Mahler, "Description of entanglement", *Physical Review A*, vol. 52, no. 6, pp. 4396, 1995.

[37] Asher Peres, "Separability criterion for density matrices", *Physical Review Letters*, vol. 77, no. 8, pp. 1413, 1996.

[38] Sam A Hill and William K Wootters, "Entanglement of a pair of quantum bits", *Physical review letters*, vol. 78, no. 26, pp. 5022, 1997.

[39] William K Wootters, "Quantum entanglement as a quantifiable resource", *Philosophical Transactions of the Royal Society of London. Series A: Mathematical, Physical and Engineering Sciences*, vol. 356, no. 1743, pp. 1717–1731, 1998.

[40] Valerie Coffman, Joydip Kundu, and William K Wootters, "Distributed entanglement", *Physical Review A*, vol. 61, no. 5, pp. 052306, 2000.

[41] Vlatko Vedral, Martin B Plenio, Michael A Rippin, and Peter L Knight, "Quantifying entanglement", *Physical Review Letters*, vol. 78, no. 12, pp. 2275, 1997.

[42] Christopher Eltschka and Jens Siewert, "Quantifying entanglement resources", *Journal of Physics A: Mathematical and Theoretical*, vol. 47, no. 42, pp. 424005, 2014.

[43] Wolfgang Dür, Guifre Vidal, and J Ignacio Cirac, "Three qubits can be entangled in two inequivalent ways", *Physical Review A*, vol. 62, no. 6, pp. 062314, 2000.

[44] Xavier Coiteux-Roy, Elie Wolfe, and Marc-Olivier Renou, "No bipartite-nonlocal causal theory can explain nature's correlations", *Physical review letters*, vol. 127, no. 20, pp. 200401, 2021.

[45] Evangelia Takou, Edwin Barnes, and Sophia E Economou, "Precise control of entanglement in multinuclear spin registers coupled to defects", *Physical Review X*, vol. 13, no. 1, pp. 011004, 2023.

[46] Nathaniel JarXiv preprint ohnston, Sarah Plosker, Charles Torrance, and Luis Varona, "Generalizing the cauchy-schwarz inequality: Hadamard powers and tensor products", *arXiv:2507.10327*, 2025.

[47] Dharmaraj Ramachandran, Aditya Dubey, Subrahmanyam SG Mantha, and Radhika Vathsan, "Robust entanglement measure for mixed quantum states: D. ramachandran et al.", *Quantum Information Processing*, vol. 24, no. 8, pp. 246, 2025.

[48] Gopal Chandra Santra, Sudipto Singha Roy, Daniel J Egger, and Philipp Hauke, "Genuine multipartite entanglement in quantum optimization", *Physical Review A*, vol. 111, no. 2, pp. 022434, 2025.

[49] Mengru Ma, Yinfei Li, and Jiangwei Shang, "Multipartite entanglement measures: A review", *Fundamental Research*, p. 1, 2024.

[50] Omar Gamel, "Entangled bloch spheres: Bloch matrix and two-qubit state space", *Physical Review A*, vol. 93, no. 6, pp. 062320, 2016.

[51] Goran Lindblad, "On the generators of quantum dynamical semigroups", *Communications in Mathematical Physics*, vol. 48, no. 2, pp. 119–130, 1976.

[52] Kurt Jacobs and Daniel A Steck, "A straightforward introduction to continuous quantum measurement", *Con-*

temporary Physics, vol. 47, no. 5, pp. 279–303, 2006.

[53] Longwen Zhou, "Entanglement phase transitions in non-hermitian kitaev chains", *Entropy*, vol. 26, no. 3, pp. 272, 2024.

[54] Yu-Guo Liu and Shu Chen, "Lindbladian dynamics with loss of quantum jumps", *Physical Review B*, vol. 111, no. 2, pp. 024303, 2025.

[55] SR Hartmann and EL Hahn, "Nuclear double resonance in the rotating frame", *Physical Review*, vol. 128, no. 5, pp. 2042, 1962.

[56] Pengcheng Yang, Martin B Plenio, and Jianming Cai, "Dynamical nuclear polarization using multi-colour control of color centers in diamond", *EPJ Quantum Technology*, vol. 3, pp. 1–9, 2016.

[57] BR Mollow, "Stimulated emission and absorption near resonance for driven systems", *Physical Review A*, vol. 5, no. 5, pp. 2217, 1972.

[58] Markus Aspelmeyer, Tobias J Kippenberg, and Florian Marquardt, "Cavity optomechanics", *Reviews of Modern Physics*, vol. 86, no. 4, pp. 1391, 2014.

[59] Eyal Buks, "Disentanglement–induced superconductivity", *Entropy*, vol. 27, no. 6, pp. 630, 2025.

[60] Shovan Dutta, Shu Zhang, and Masudul Haque, "Quantum origin of limit cycles, fixed points, and critical slowing down", *Physical Review Letters*, vol. 134, no. 5, pp. 050407, 2025.

[61] Ching-Kit Chan, Tony E Lee, and Sarang Gopalakrishnan, "Limit-cycle phase in driven-dissipative spin systems", *Physical Review A*, vol. 91, no. 5, pp. 051601, 2015.

[62] Haggai Landa, Marco Schiró, and Grégoire Misguich, "Correlation-induced steady states and limit cycles in driven dissipative quantum systems", *Physical Review B*, vol. 102, no. 6, pp. 064301, 2020.

[63] VS Lvov and LA Prozorova, "Spin waves above the threshold of parametric excitations", *Spin waves and magnetic excitations*, vol. 239, pp. 233, 1988.

[64] Nicolas Roch, Serge Florens, Vincent Bouchiat, Wolfgang Wernsdorfer, and Franck Balestro, "Quantum phase transition in a single-molecule quantum dot", *Nature*, vol. 453, no. 7195, pp. 633–637, 2008.

[65] L Thomas, FL Lionti, R Ballou, Dante Gatteschi, Roberta Sessoli, and B Barbara, "Macroscopic quantum tunnelling of magnetization in a single crystal of nanomagnets", *Nature*, vol. 383, no. 6596, pp. 145–147, 1996.

[66] Sergey Trishin, Christian Lotze, Nils Bogdanoff, Felix von Oppen, and Katharina J Franke, "Moiré tuning of spin excitations: Individual fe atoms on mos 2/au (111)", *Physical Review Letters*, vol. 127, no. 23, pp. 236801, 2021.

[67] GG Blesio and AA Aligia, "Topological quantum phase transition in individual fe atoms on mos 2/au (111)", *Physical Review B*, vol. 108, no. 4, pp. 045113, 2023.

[68] Tomoaki Yamasaki, Miki Ueda, and Satoru Maegawa, "A hysteresis phenomenon in nmr spectra of molecular nanomagnets fe8: a resonant quantum tunneling system", *Physica B: Condensed Matter*, vol. 329, pp. 1187–1188, 2003.

[69] S Venkataramani, U Jana, M Dommaschk, FD Sönnichsen, F Tuczek, and R Herges, "Magnetic bistability of molecules in homogeneous solution at room temperature", *Science*, vol. 331, no. 6016, pp. 445–448, 2011.

[70] Martin Koppenhöfer, Christoph Bruder, and Alexandre Roulet, "Quantum synchronization on the ibm q system", *Physical Review Research*, vol. 2, no. 2, pp. 023026, 2020.

[71] Chao Yin, Federica M Surace, and Andrew Lucas, "Theory of metastable states in many-body quantum systems", *Physical Review X*, vol. 15, no. 1, pp. 011064, 2025.

[72] Daniel Manzano, "A short introduction to the lindblad master equation", *Aip Advances*, vol. 10, no. 2, pp. 025106, 2020.

[73] Howard Carmichael, *An open systems approach to quantum optics: lectures presented at the Université Libre de Bruxelles, October 28 to November 4, 1991*, vol. 18, Springer Science & Business Media, 2009.

[74] Sean M Carroll and Ashmeet Singh, "Quantum mereology: Factorizing hilbert space into subsystems with quasiclassical dynamics", *Physical Review A*, vol. 103, no. 2, pp. 022213, 2021.

[75] Arthur J Parzygnat, Tadashi Takayanagi, Yusuke Taki, and Zixia Wei, "Svd entanglement entropy", *Journal of High Energy Physics*, vol. 2023, no. 12, pp. 1–45, 2023.

[76] Jens Siewert, "On orthogonal bases in the hilbert-schmidt space of matrices", *Journal of Physics Communications*, vol. 6, no. 5, pp. 055014, 2022.

[77] Lemin Lai and Shunlong Luo, "Steerability criteria based on heisenberg–weyl observables", *Journal of Physics A: Mathematical and Theoretical*, vol. 56, no. 11, pp. 115305, 2023.

[78] Xiaofen Huang, Tinggui Zhang, Ming-Jing Zhao, and Naihuan Jing, "Separability criteria based on the weyl operators", *Entropy*, vol. 24, no. 8, pp. 1064, 2022.